Global Structure of Micro-instabilities in Tokamak Plasmas: Stiff Transport or Plasma Eruptions?
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(Dated: April 4, 2013)

Solutions to a model global 2D eigenmode equation describing micro-instabilities in tokamak plasmas are presented that demonstrate a sensitivity of the mode structure and stability to plasma profiles. In narrow regions of parameter space, with special plasma profiles, a maximally unstable mode is found that balloons on the outboard side of the tokamak. This corresponds to the conventional picture of a ballooning mode. However, for most profiles this mode cannot exist and instead a more stable mode is found that balloons closer to the top or bottom of the plasma. Good quantitative agreement with a 1D ballooning analysis is found provided the constraints associated with the higher order profile effects, often neglected, are taken into account. A sudden transition from this general mode to the more unstable ballooning mode can occur for a critical flow shear, providing a model for why some experiments observe small plasma eruptions (Edge Localised Modes, or ELMs) in place of large Type I ELMs.

Micro-instabilities in magnetised plasmas are those with a characteristic length scale across magnetic field lines comparable to the ion Larmor radius, $\rho_i$. The particle drifts play a key role in the dynamics, and therefore often characterise their growth rate, which is weak compared to Alfvénic modes. While these micro-instabilities are relatively benign they are nevertheless important because they drive turbulence that degrades plasma confinement. Understanding turbulence and its influence on confinement is one of the key challenges facing magnetically confined plasmas, especially fusion plasmas.

In this Letter, we consider the generic linear properties of the micro-instabilities that drive turbulence in a toroidal magnetic confinement device, such as a tokamak. Our new calculations of their global eigenmode structure are compared with analytic ‘ballooning formalism’ approaches that reduce the problem to 1D [1]. These 1D models have identified two types of mode structure: (1) an isolated mode is the most unstable and is related to the conventional ballooning mode of ideal magnetohydrodynamics (MHD) [2–5], but only exists in very special circumstances, and (2) a general mode which is more stable, but exists throughout the plasma. We find both of these mode structures in our 2D global mode calculations, providing first quantitative tests that confirm the ballooning theory. Our results suggest a possible mechanism for a sudden transition from benign micro-instabilities that drive turbulence (perhaps clamping gradients in a stiff transport model) to stronger instabilities that could release small filamentary eruptions with an associated collapse of the profiles. This mechanism may form the basis of a model for small Edge Localised Modes (ELMs) in tokamaks [9], potentially preventing the plasma gradients from building sufficiently to drive ideal MHD modes that are believed to be responsible for large Type I ELMs.

While our results are generic to any micro-instability in a tokamak plasma, it is helpful to illustrate the analysis with a particular model. Thus, we consider in a large aspect ratio, circular cross section tokamak equilibrium, with toroidal flux surfaces labelled by the minor radius coordinate, $r$. We restrict consideration to electrostatic fluctuations, and adopt an adiabatic response for the electrons. For the ions, our starting point is the gyrokinetic equation for the ion distribution function, which we expand assuming that the effects associated with the parallel ion dynamics and toroidal drifts are small compared to the mode frequency. If we also consider a wavelength across magnetic field lines that is somewhat larger than the ion Larmor radius, quasi-neutrality provides the following equation for the perturbed electrostatic potential, $\phi$ [6]:

\[
\left[ \rho_i^2 \frac{\partial^2}{\partial x^2} - \frac{\sigma^2}{\omega^2} \left( \frac{\partial}{\partial \theta} + ik_\theta x \right)^2 - \frac{2\epsilon_n}{\omega} \left( \cos \theta + \frac{i \sin \theta}{k_\theta} \frac{\partial}{\partial x} - \frac{\omega - 1}{\omega + \eta_i} - k_\theta^2 \rho_i^2 \right) \right] \phi(x, \theta) = 0
\]  

(1)

where $\sigma = \epsilon_n/(qk_\theta \rho_i)$, $\epsilon_n = L_n/R$, $L_n$ is the density gradient scale length, $R$ is the major radius, $\omega$ is the complex mode frequency normalised to the electron diamagnetic frequency, $k_\theta$ is the poloidal wave number, $\eta_i$ is the ratio...
of density to ion temperature gradient length scales and we have assumed equal electron and ion temperatures. The poloidal angle, \( \theta \) is defined so that \( \theta = 0 \) is at the outboard mid-plane and \( x = r - r_s \) is the distance from the rational surface where \( r = r_s \). The model is valid in the core relevant limit \( \eta_s \gg 1 \) (which then recovers the ion temperature gradient, ITG, mode) or \( \omega \simeq 1 \) (which then recovers the electron drift wave). Our focus here will be on the ITG mode.

The 2D eigenmode equation can be conveniently solved by first adopting a Fourier transform representation of the perturbed electrostatic field [7]:

\[
\phi(x, \theta) = \int_{-\infty}^{\infty} A(\theta_0) \xi(\theta, \theta_0) \exp\left[i n q' x (\theta_0 - \theta)\right] d\theta_0 \tag{2}
\]

providing an eigenvalue condition that relates \( \omega, \theta_0 \) and \( x \). We write this relationship in the form \( \omega = \Omega(x, \theta_0) \). The dependence of the function \( \Omega \) on \( x \) and \( \theta_0 \) is obtained by solving eq. 3 numerically over all \( \theta_0 \) for the range of flux surfaces of interest, noting that the various equilibrium parameters in the equation vary slowly with \( x \). If we restrict consideration to small \( x \) (anticipating modes radially localised about \( r = \rho_s \)), we can Taylor expand \( \Omega = \omega_0(\theta_0) + \omega_x(\theta_0) x + \left[\omega_{xx}(\theta_0)/2\right] x^2 + \cdots \). Transforming into the Fourier space, it is straightforward to show that \( x \phi \) maps to \((i/nq') A/d\theta_0\) and \( x^2 \phi \) maps to

\[
-1/(nq')^2 d^2 A/d\theta_0^2,
\]

so that \( nq' \theta_0 \) can be interpreted as a radial wave-number at \( \theta = 0 \) (\( n \) is the toroidal mode number and \( q' = dq/dr \) where \( q \) is the safety factor). Following a procedure which extends that set out in Ref [1] we can then reduce our 2-D partial differential equation to a sequence of 1-D ordinary differential equations by expanding in \( n \), assumed to be large. The amplitude factor \( A(\theta_0) \) is assumed to vary faster with \( \theta_0 \) than \( \xi(\theta, \theta_0) \) does. Substitution of eq. 2 into eq. 1 yields to leading order the ballooning equation for \( \xi(\theta, \theta_0) \):

\[
\frac{\sigma^2}{\omega^2} \frac{d^2}{dx^2} + \frac{k_d^2 \rho_s^2 s^2 (\theta - \theta_0)^2}{\omega} + 2 \epsilon_n \frac{\cos \theta + s (\theta - \theta_0) \sin \theta}{\omega + \eta_s} - k_d^2 \rho_s^2 \xi(\theta, \theta_0) = 0 \tag{3}
\]

where the sign of the square root is chosen to give a bounded solution in \( \theta \). Note that \( \Lambda \) has a width \( \sim n^{-1/2} \), justifying our Taylor expansion of \( \omega_0(\theta_0) \) about \( \theta_0 = \theta_m \).

The eigenvalue condition provides:

\[
\omega = \omega_0(\theta_0 = \theta_{m}) - \left(\omega_{xx}(\theta_0) \theta_0 \right)^{1/2} + \mathcal{O}(n^{-2}) \tag{4}
\]

Note that the isolated mode growth rate is determined from the 1-D ballooning equation by simply evaluating the eigenvalue at the \( x \) and \( \theta_0 \) values which maximise the growth rate. Furthermore, if \( A(\theta_0) \) is highly localised around \( \theta_0 = \theta_m \), the Fourier transform of eq. 2 will be dominated by the region around \( \theta_0 = \theta_m \). This, together with the fact that \( \xi(\theta, \theta_0) \) peaks close to \( \theta = \theta_0 \), leads to an expression for the potential, \( \phi \) which peaks about the poloidal angle \( \theta = \theta_m \). For our up-down symmetric model, where \( \theta_m = 0 \), we predict that there will be an isolated mode localised on the outboard side of the tokamak.

To summarise, the isolated mode exists when \( \omega_x = 0 \); one selects the value of \( \theta_0 \) to maximise the growth rate, and the mode is localised about a poloidal angle \( \theta \) equal to that value of \( \theta_0 \), which is often the outboard mid-plane. While this is intuitive from a physics point of view, if \( \omega_x \neq 0 \) (which is usually the case), the constraints of the higher order theory do not allow such a mode to exist, as we now discuss.

In general, the first order derivative of eq. 4 cannot be neglected. Indeed, the linear term dominates over the quadratic one for \( \omega_x \gtrsim \omega_{xx}/n^{-1/2} q' \). Often \( \omega_x \) will be a complex variable, so to neglect this term its real and imaginary parts must vanish at the same value of \( x \); hence the isolated mode can only exist under very special
situations. In the more general case we can neglect the term of eq. 4 in $\omega_{xx}$. Dividing the remaining terms by $A\omega_x$ and integrating over a full period in $\theta_0$, we derive the eigenvalue condition:

$$\omega = \langle \omega_x \omega_x^{-1} \rangle / \langle \omega_x^{-1} \rangle$$

where angled brackets denote an average of the enclosed quantity over $\theta_0$. Substituting this value of $\omega$ into eq. 4 and integrating yields the required periodic expression for $A(\theta_0)$. One finds that it is still localised in $\theta_0$ provided $\Omega(x, \theta_0)$ is complex, but now at the value of $\theta_0$ close to where $\omega = \omega_0(\theta_0)$; this position is typically at the top or bottom of the tokamak, corresponding to $\theta_0 = \pm \pi/2$.

This general mode, then, is more stable than the isolated mode, having a growth rate which is the average of $\omega_0$ over $\theta_0$ rather than the maximum (resulting in a higher critical gradient than the isolated mode), and sits close to the top or bottom of the plasma rather than the outboard mid-plane.

To provide quantitative results, we solve eq. 3 for a specific parameter set: $n = 50$, $\delta = 2$, $k\delta \rho_i = 0.33$, $R/a = 10$. Focusing on a mode centred on the rational surface $q(r = r_s) = 1.8$ a locally quadratic $q$-profile is employed with $q(r) = 3.45(r/a)^2$. We adopt a further simplification that all parameters in eq. 1 are independent of $x$ except for $\eta$. This ensures that if $\eta$ has a maximum in $x$, an isolated mode will exist there (as we shall illustrate shortly), while a linear $\eta$ profile is expected to yield a general mode. We fix $\eta = 5$ at $r = r_s$ resulting in the same local eigenvalue $\Omega(x = 0, \theta_0)$ in all cases. While the $q$-profile is not taken into account in the model parameters, it is required to provide a distribution of rational surfaces across the minor radius.

To provide quantitative tests of this generalised ballooning theory, we provide full global solutions to the 2D model eq. 1. We first decompose $\phi(x, \theta)$ into poloidal Fourier harmonics, $\phi(x, \theta) = \sum m u_m(x) \exp(i m \theta)$, and then solve the set of coupled equations for the radial dependence of the Fourier coefficients, $u_m(x)$, determining the complex mode frequency, $\omega$, as a global eigenvalue of the system (the subscript $m$ labels the Fourier harmonic). The results for the two different eigenmode structures are shown in figure 2. One can immediately see that for the peaked $\eta$ profile the mode is indeed localised on the outboard side, while for the linear profile case it is peaked at the top of the plasma. We find no global eigenmode on the outboard side when the $\eta$ profile is linear. Also shown is the radial dependence of each of the Fourier harmonics. Figure 2(b) shows the classic ballooning mode structure, with all Fourier harmonics combining in phase (i.e. $\theta_0 = 0$) with a Gaussian envelope. The situation for the general mode, shown in figure 2(d) is more complex, reflecting the involvement of a range of $\theta_0$. These mode structures can also be evaluated from the 1D ballooning procedure; we do not show them here because to the eye they are indistinguishable from these global solutions. A quantitative test is to compare the complex mode frequencies. The two values of global eigenvalue we obtain for the 2D analysis are $\omega = -0.025 + i0.319$ and $\omega = -0.110 + i0.239$ for the peaked and linear $\eta$ profiles.
which poloidal cross-section shows the simulation domain. The existence of the Fourier transform, eq. \( \omega \) is periodic in \( \theta \), and the \( A \) function of the poloidal harmonics, \( u_m(x) \), for the peaked and linear \( \eta \) profiles respectively. The (orange) shaded region in the poloidal cross-section shows the simulation domain.

In Ref [8] Kim and Wakatani argued that a continuum of modes can exist when \( \omega_x \neq 0 \). However, these are not eigenmodes of the system. Such a continuum arises because there is a range of solutions to eq. 4 which provide the desired localisation of \( A(\theta_0) \) and hence the existence of the Fourier transform, eq. 2. However, in general, those solutions do not provide a form for \( A \) that is periodic in \( \theta_0 \), which is required for \( \phi \) to be periodic in poloidal angle, \( \theta \). For this reason, the Kim-Wakatani modes are not physical eigenmodes, and we see no sign of them in our global solutions.

We can use our global solutions to explore the relation between isolated and general modes. We start with an \( \eta \) profile which is peaked at the rational surface labelled by \( x = 0 \) so that the more unstable isolated mode exists. Adding a linear contribution to the \( \eta \) profile simply shifts the position of the maximum of \( \eta \); the isolated mode still exists, but adjusts its position to sit at the point where \( \eta \) is a maximum. A more interesting situation arises when one introduces flow shear into the problem as this then shifts the position where the local frequency is stationary relative to that where the local growth rate peaks. No isolated mode is then possible. Thus in eq. 1 we introduce a Doppler shift \( \omega \to \omega + n\Omega_x \), working in the rest frame of the rational surface. We parameterise the toroidal flow shear, \( \Omega' \), in the shear rate parameter \( \gamma_E = d\Omega/dq \). Figure 3(a,b) shows how the mode frequency and growth rate respond to the flow shear; they are symmetric under \( \gamma_E \to -\gamma_E \). Note how the isolated mode that exists at \( \gamma_E = 0 \) smoothly evolves into the general mode for a relatively low value of \( |\gamma_E| \sim 0.015 \ll \gamma \), where \( \gamma \) is the ITG growth rate [10]. Thus the window of existence for the isolated mode is very small. We also show in figures 3(c,d) how the eigenmode structure in the poloidal plane evolves from the outboard mid-plane for \( \gamma_E = 0 \) (figure 2(a)) through to the top of the plasma for \( \gamma_E \gtrsim 0.01 \). With a negative value of \( \gamma_E \) the mode moves towards the top of the plasma. If we start with a general mode (i.e. linear \( \eta \) profile) and add flow shear, there is actually only a small impact on the stability, which can be understood in terms of our 1D ballooning analysis. The radial width of the mode is affected because of its dependence on \( \omega_x \).

In conclusion, we have presented global 2D eigenmode solutions for micro-instabilities in toroidal geometry. These confirm that the often used approach to ballooning theory that assumes one can select the value of \( \theta_0 \) to maximise the growth rate, giving rise to a mode that balloons on the outboard side, is only valid in very special situations. More generally, one should average the local (ballooning) growth rate over \( \theta_0 \), and the mode will exist near the top or bottom of the plasma. Our 2D calculations for a representative model of toroidal drift waves demonstrate good quantitative agreement between the global eigenmode frequency and that obtained from the 1D ballooning theory for both types of mode structure, provided one treats \( \theta_0 \) correctly. Considering the effect of flow shear, we have shown that an isolated mode smoothly transforms into the more stable general mode as the flow shear is increased, and used this study to quantify the very narrow region of parameter space that the isolated mode occupies.

Noting that the results outlined here are generic for any local micro-instability model in a tokamak plasma we close with a discussion of how our results may help
to explain small ELM regimes in tokamaks. More generally, these results may explain why sometimes plasma gradients can be clamped by micro-instabilities through a steady diffusive process driven by the associated turbulence (‘stiff’ transport), whereas in other situations such instabilities can lead to a more bursty energy release (such as small ELMs). In the high confinement H-mode, steep density and temperature gradients form in the so-called pedestal region at the tokamak plasma edge. This would lead to a maximum in the linear drive within this pedestal region, and therefore a maximum in the local linear growth rate, $\Im (\Omega)$, there. However, depending upon the relevant instability model, the local mode frequency, $\Re (\Omega)$, would not in general be stationary at the same position. The isolated mode would then not be accessible to the plasma, and the relatively stable general mode would provide the drive for the turbulence, constraining the gradients at relatively high values compared to the critical gradients for instability to the isolated mode. This pedestal region is then susceptible to large ELMs (so-called Type I ELMs) driven by the ideal MHD peeling-ballooning mode \cite{11–13} when the pressure gradient reaches a threshold value to drive this instability. These ELMs periodically collapse the profiles in the pedestal, which then rebuild to trigger the next one. As the profiles rebuild between ELMs, the position where $\Re (\Omega)$ is stationary will alter and at some time during the pedestal evolution this position may briefly align with that where $\Im (\Omega)$ is also stationary, allowing an isolated mode to exist. This would be highly unstable, causing a crash in the gradients (i.e. a small ELM) on the timescale of the rapid transition between modes, that takes them away from the ideal MHD instability boundary associated with large Type I ELMs.

The authors gratefully acknowledge helpful discussions with Bryan Taylor and Jack Connor. This work was part-funded by the RCUK Energy Programme under grant EP/I501045 and the European Communities under the contract of Association between EURATOM and CCFE. To obtain further information on the data and models underlying this paper please contact PublicationsManager@ccfe.ac.uk. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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