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Abstract

It is shown that if the flow vector is a coordinate basis vector, then perfect dynamo action is not possible, regardless of whether the steady flow is compressible. Criteria for determining the basis vector property are found to be expressible in terms of Lie derivatives that are straightforward to compute.

1. Introduction

The magnetic induction equation for compressible flow may be formulated in terms of a Lie derivative of a vector by introducing the field defined as the the magnetic field $\mathbf{B}$ divided by the mass density, a result originally due to Walen [1, § 4-2], which is rederived for example in Chandrasekhar [2, § 38],

$$d\tilde{\mathbf{B}}/dt = \mathcal{L}_{\mathbf{u}}(\tilde{\mathbf{B}}) \tag{1}$$

where $\mathcal{L}_{\mathbf{u}}$ is the Lie derivative with respect to the flow field $\mathbf{u}$, $\tilde{\mathbf{B}} = \mathbf{B}/\rho$ and ρ is mass density.

This work focusses on the perfect dynamo problem rather than the fast dynamo, see Section 2.1 for a discussion as to how they are interrelated. The perfect problem may be posed in terms of solutions of Eq. (1), namely the flow $\mathbf{u}$ is said to be a perfect dynamo action if volume-integrated absolute fluxes of $\mathbf{B}$ grows exponentially in time for some seed field [3].

At the root of the current work is the simple observation [4] that if the vector $\mathbf{u}$ in the Lie derivative $\mathcal{L}_{\mathbf{u}}(\tilde{\mathbf{B}})$ may be identified as a basis vector $\mathbf{e}_3$ in a coordinate basis, then Eq. (1) reduces to pure advection, viz.

$$\partial \tilde{B}^i/\partial t = \partial_3 \tilde{B}^i \tag{2}$$

where $\tilde{B}^i$ are the contravariant components of $\tilde{\mathbf{B}}$ and ∂_3 is a shorthand for $\partial/\partial\bar{x}^3$. This immediately implies that $\tilde{B}^i$ is conserved following the flow and, as discussed in Section 2, rules out perfect dynamo action for all flows in the aforementioned class.

Hence, this generalises the known antidynamo results that require invariance in one Cartesian coordinate, see [5, Chap. V],[6]. It is to be compared to the result that a perfect dynamo action requires a specific amount of topological entropy [7]. This property is not obvious to compute whereas the properties of coordinate basis vectors are simply described, and it is a property which it is easy to test in a particular $\mathbf{u}$. Remarkably, the principal test relies on establishing a property for the basis vectors which involves Lie brackets, equivalent entities to Lie derivatives of vectors, namely that Lie brackets of the basis vectors vanish.

The result established herein is in some respects not as general as that of [7] viewed as an anti-dynamo theorem for flows with zero topological entropy, but in other respects, it represents an important extension in that a class of compressible flows is now also excluded.

2. Mathematics

2.1. Dynamo Definitions

Fast dynamo action is defined in the context of the 'classic' version of the magnetic induction equation with resistivity ϵ, viz.

$$d\mathbf{B}/dt = \nabla \times (\mathbf{u} \times \mathbf{B}) + \epsilon \nabla^2 \mathbf{B} \tag{3}$$

If the magnetic energy of solutions to Eq. (3) grows exponentially with positive growth rate in the limit $\epsilon \to 0$ (for some initial seed field), then the flow $\mathbf{u}(\mathbf{x})$ is said to be a fast dynamo. The results presented herein concern the case $\epsilon = 0$, $\mathbf{u}(\mathbf{x})$ is a perfect dynamo if fluxes of magnetic field grow exponentially, strictly that the following inequality be satisfied

$$\mathrm{Lt}_{t\to\infty} \frac{1}{t} \log \left(\int_V |\mathbf{B}(\mathbf{x}, t)| dV \right) > 0 \tag{4}$$

The two dynamo problems are physically quite different in that nonzero ϵ is more realistic as even tiny amounts of dissipation prevent the appearance of field singularities. The two are clearly related however, for it is reasonable to conjecture that since diffusion only acts to weaken dynamo action, if a flow cannot be a perfect dynamo, it cannot be a fast dynamo either. Ref [7] successfully establishes the reverse implication, namely that the topological entropy criterion for perfect dynamo action also ensures fast dynamo action.

2.2. Geometry

A set of three vectors $\{\mathbf{e}_i,\ i=1,2,3\}$, forms a basis in 3-D provided the vectors are linearly independent at each point. The vectors are said to form a coordinate basis if each may be parameterised by $\bar{x}^i$ such that the $\bar{x}^i$ may be used as a set of coordinates. Thus a coordinate basis is determined by a mapping from parameter space to real space $\mathbf{x}(\bar{x}^1,\bar{x}^2,\bar{x}^3)$ such that

$$\mathbf{e}_i = \partial\mathbf{x}/\partial\bar{x}^i, \quad i=1,2,3 \tag{5}$$

form a basis.

Now the in any reasonable 3-D coordinate system, there is the remarkable result that the Lie derivative of a vector may be written

$$\mathcal{L}_{\mathbf{u}}(\mathbf{v})^i = v^k\frac{\partial u^i}{\partial x^k} - u^k\frac{\partial v^i}{\partial x^k} \tag{6}$$

Hence the Lie bracket notation as an equivalent for the Lie derivative

$$\mathcal{L}_{\mathbf{u}}(\mathbf{v}) = [\mathbf{u},\mathbf{v}] \tag{7}$$

so that the two vectors $\mathbf{u}$ and $\mathbf{v}$ appear on an equal footing. Adopting this notation [8, § 4.5], it may be shown the $\mathbf{e}_i$ defined by Eq. (5) satisfy

$$[\mathbf{e}_i,\mathbf{e}_j] = \mathbf{0}, \quad \forall i,\ j \tag{8}$$

and that if Eq. (8) holds, linearly independent vectors $\{\mathbf{e}_i,\ i=1,2,3\}$ form a coordinate basis. (Strictly speaking, this last statement requires the Poincare lemma, which applies in toroidal geometry only if the toroidal angles are allowed free range, ie. not restricted to $[0,2\pi]$.) Note that $\{\lambda(\mathbf{x})\mathbf{e}_i\}$ is not in general a coordinate basis unless $\lambda=$ const. $\neq 0$, for example

$$[\lambda\mathbf{e}_1,\lambda\mathbf{e}_2] = \lambda^2[\mathbf{e}_1,\mathbf{e}_2] + \lambda\left(\mathbf{e}_2\partial\lambda/\partial\bar{x}^1 - \mathbf{e}_1\partial\lambda/\partial\bar{x}^2\right) \tag{9}$$

To establish anti-dynamo action, it is necessary to show that $\mathbf{u}$ may be expressed as a coordinate basis vector. This is equivalent to showing that there exists a mapping $\mathbf{x}(\bar{x}^1,\bar{x}^2,\bar{x}^3)$ such that

$$\mathbf{u} = \partial\mathbf{x}/\partial\bar{x}^3 \tag{10}$$

for some coordinates $\bar{x}^i$, and $\mathbf{u}\neq\mathbf{0}$. For example, $\bar{x}^3$ could be a function of arc length along a streamline. To complete the coordinate system, streamline labels in a plane normal to $\mathbf{u}$ (cf. Clebsch variables) could be used. Note that the possibility of such a coordinate basis system depends on the flow's having a relatively simple topology. Regardless, the minimal requirement for the other two vectors say $\mathbf{e}'_i$, $i = 1, 2$ in the frame is that $[\mathbf{e}_3, \mathbf{e}'_i] = 0$.

Limiting the applicability of many of the results which follow is the Poincare-Hopf theorem relating the number of zeroes of a vector field to the topology of the compact manifold on which it is defined (which gives the 'hairy-ball' theorem in the case of spherical surfaces). For a vector field to form part of a basis, it is obviously necessary that it be non-zero everywhere, implying that the only compact coordinate systems are to be found in a toroidal geometry. Thus, apart from toroidal systems, all the dynamo results in this section apply only to unbounded flows.

2.3. Application

The anti-dynamo result outlined in Section 1 requires further discussion. Using the results of the previous section, Eq. (2) is derived as follows. Writing $\tilde{\mathbf{B}} = \tilde{B}^i \mathbf{e}_i$, and substituting in Eq. (1), using Eq. (8), gives

$$d\tilde{\mathbf{B}}/dt = \mathbf{e}_i \partial \tilde{B}^i / \partial \bar{x}^3 \tag{11}$$

Taking components gives Eq. (2). Although the $\tilde{B}^i$ are simply advected from place to place, the physical magnetic field is given by $B^i \mathbf{e}_i$ (summation convention applies), so will change over time according as the basis vector $\mathbf{e}_i$ changes with position. Similarly, since it is actually $\tilde{\mathbf{B}} = \mathbf{B}/\rho$ which is conserved, there is a further factor due to the change in the volume element. This may be seen from the equation for conservation of mass $m = \rho\sqrt{g}$, which is also expressible in terms of a Lie derivative and so reduces to

$$\partial m/\partial t = \partial_3 m \tag{12}$$

However, for a coordinate system without singularities, derived geometric quantities such as $\mathbf{e}_i$ and $\sqrt{g}$ are well-behaved, in particular $\sqrt{g} \neq 0$. Thus although the physical $\mathbf{B}$ changes, an exponential increase without limit is ruled out.

The absence of singularities is important as it rules from consideration such situations as a purely radial inflow. Either the radial flow continues to the coordinate singularity or it is stopped on some surface. But if $\mathbf{u} = \mathbf{0}$ at any point, then it cannot be part of a basis there.

2.4. Time dependent flows

In the case of time dependent flows, an antidynamo result might be established for a $\mathbf{u}$ which is one of a set of time varying coordinate basis vectors, only if further conditions are placed on the basis variation. Compressibility also changes the mass conservation equation because m only evolves as a Lie derivative if $\sqrt{g}$ is time invariant. Hence, with $\mathbf{e}_i(t)$ the appropriate evolution equations for magnetic field and density are

$$d\tilde{B}^i/dt + \tilde{B}^j \mathbf{e}^i \cdot d\mathbf{e}_j/dt = \partial_3 \tilde{B}^i \tag{13}$$

$$\partial\rho/\partial t = (1/\sqrt{g})\partial_3(\sqrt{g}\rho) = \partial_3\rho + \rho\partial_3 \log\sqrt{g} \tag{14}$$

Thus in each equation there is an additional term which may cause growth, unless quite specific constraints are placed on the metric. Since the explicit time variation of the basis vectors appearing in Eq. (13) may be avoidable, whereas the term in $\sqrt{g}$ may not, there follows the implication that an incompressible, time dependent flow is less likely to be a perfect dynamo than a compressible time dependent flow.

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