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The absorption of electron cyclotron waves in the vicinity of an extremum of the equilibrium magnetic field

C. N. Lashmore-Davies, R. O. Dendy, and R. J. Hastie
Culham Laboratory, Abingdon, Oxfordshire OX14 3DB, England (Euratom/UKAEA Fusion Association)

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Local extrema in the equilibrium magnetic field strength can occur in tokamak plasmas when the plasma beta is increased. Furthermore, the magnetic field strength experienced by an electromagnetic wave propagating along a vertical chord in lower-beta tokamak plasmas may also pass through an extremum. For both these reasons, it is of interest to calculate the absorption of electron cyclotron waves in configurations where a harmonic of the cyclotron resonance lies near an extremum of the equilibrium magnetic field strength. It is shown that strong absorption can occur for both the O mode at second harmonic resonance and the X mode at third harmonic resonance, whereas absorption at these resonances is relatively weak when the gradient in magnetic field strength is linear. The optical depth in a magnetic field with parabolic scale length \( L \) is typically enhanced with respect to that in a field with linear scale length \( R \) by a factor \( cL / 5v_T R \), where \( v_T \) is the electron thermal velocity and \( c \) the velocity of light. Since electron cyclotron waves can reach these resonances in high-density plasma from launch positions on the low-field side of the tokamak, the number of effective and convenient electron cyclotron heating configurations is increased when the equilibrium magnetic field strength has local extrema.

I. INTRODUCTION

The absorption of electromagnetic waves at the electron cyclotron resonance (or one of the harmonics) depends on the spatial variation of the equilibrium magnetic field.\(^1\) For a tokamak plasma this is usually a simple monotonic dependence because of the toroidal field coils. The slower the spatial variation, the longer a particle remains in resonance and, therefore, the stronger the absorption. The scale length in the normal situation is the tokamak major radius. However, the equilibrium field strength can exhibit an extremum, for example, when the plasma-beta increases or for propagation along a vertical chord. Since the magnetic field varies more slowly in the neighborhood of an extremum, it is interesting to consider whether cyclotron absorption will be enhanced in such a region. This problem was first considered by Chu and Hui\(^2\) in the context of electron cyclotron heating at a field minimum by the ordinary mode (O mode) and extraordinary mode (X mode) at first and second harmonic resonance. Thus detailed information on absorption profiles in real space and velocity space already exist for these cases, whose application in Ref. 2 was to tandem mirrors.

For electron cyclotron waves in present tokamak conditions, the absorption is already strong in linear field gradients for the fundamental resonance and for the second harmonic of the X mode; see, for example, Refs. 1, 3, and 4. On the other hand, the O-mode second harmonic and the X-mode third harmonic are only weakly absorbing. An enhancement in the absorption would therefore be more significant for these latter two modes. In addition, the O-mode second harmonic and the X-mode third harmonic would give accessibility to higher-density tokamak plasmas, from a low-field side launch position. We shall therefore extend the work in Ref. 2 in two respects. First, as already indicated, we shall consider the absorption of the third harmonic X mode (a case not analyzed in Ref. 2) and the second harmonic O mode at a field minimum and a field maximum. Second, we shall also consider the consequences of tuning the wave frequency just below or just above the local extremum. Chu and Hui\(^2\) calculated optical depths for general magnetic field profiles but restricted themselves to the case where the frequency was tuned to the local extremum. Interestingly, we find that the absorption is strongest when the frequency is tuned just below the local extremum and not at the extremum itself. The reason for this behavior is that when \( \omega - \Omega_0 < 0 \) (where \( \Omega_0 \) is the electron cyclotron frequency at the extremum calculated using the rest mass) the lowest energy electrons are excluded from resonance. Since the optical depth at these harmonics of the electron cyclotron frequency takes its maximum value for perpendicular propagation,\(^1\) we shall confine our analysis to this case, in which absorption is due to the relativistic mass shift. Let us first consider some of the qualitative features of the relativistic resonance condition before embarking on a detailed calculation of the absorption.

II. THE RESONANCE CONDITION

We shall consider the equilibrium magnetic field

\[
B_0(x) = i_x B_0(1 \pm x^2/L^2),
\]

(1)

corresponding to a minimum or maximum of the magnetic field at \( x = 0 \). Here \( L \) is the scale length of the field in the vicinity of the extremum; we shall include a brief discussion of this quantity later in the paper.

For perpendicular propagation, the condition for cyclotron resonance at the \( l \)th harmonic is

\[
\omega - l\Omega(x)/\gamma = 0,
\]

(2)

where
\[ \gamma = (1 - \frac{v^2}{c^2})^{-1/2}. \] (3)

Assuming \( v^2/c^2 \ll 1 \) and using Eq. (1), the condition for cyclotron resonance in the vicinity of the extremum at \( x = 0 \) is
\[ \omega - \Omega_0(x)(1 - \frac{2v^2}{c^2}) = 0, \] (4)
where
\[ \Omega_0(x) = \Omega_0(1 + \frac{x^2}{L^2}). \] (5)
Here \( \Omega_0 \) is the cyclotron frequency calculated using the rest mass and is taken as a positive quantity. Since \( x^2 \ll L^2 \), we obtain
\[ \omega - \Omega_0 \approx \Omega_0 \frac{x^2}{L^2} + \frac{\Omega_0 v^2}{2c^2} = 0. \] (6)

Defining
\[ \Delta \omega = \omega - \Omega_0, \] (7)
the resonance condition may finally be written as
\[ 2 \frac{\Delta \omega}{\Omega_0} + 2 \frac{x^2}{L^2} + \frac{v^2}{c^2} = 0. \] (8)

It is well known that, in the limit \( L \to \infty \), the relativistic resonance condition can only be satisfied when \( \Delta \omega < 0 \) for perpendicular propagation. With the aid of Eq. (8) we now distinguish three cases, which are illustrated in Figs. 1 and 2.

First, consider case (a):
\[ \omega - \Omega_0 > 0. \]

For the field maximum, this means that the wave frequency is everywhere greater than the cyclotron frequency and hence there is no absorption. However, for a field minimum,

there are two cold plasma resonances symmetrically placed on either side of the minimum. No absorption occurs in the region between the two cold resonances, where the field variation is at its weakest, but absorption occurs symmetrically outside this central region. In case (b) the cold plasma resonance is tuned to the extremum position:
\[ \omega - \Omega_0 = 0. \]

The resonance condition can be satisfied for a field minimum but not for a field maximum, since in the latter case both sides of the cold plasma resonance are low-field regions. Hence we expect absorption on both sides of the field minimum and no absorption on either side of a field maximum. Finally, in case (c),
\[ \omega - \Omega_0 < 0. \]

In this situation, the resonance condition can be satisfied for a field maximum as well as a field minimum. For a field minimum, the resonance condition can be satisfied right across the extremum region. However, since \( \Delta \omega \) is now negative, low-energy electrons cannot come into resonance with the wave and absorption occurs selectively on thermal and higher energy electrons. For the field maximum, there are two cold plasma resonances on either side of the extremum position, and absorption occurs only in the region between these two cold resonances. In the vicinity of the maximum, where \( |\Delta \omega| \) is largest and the field variation weakest, low-energy electrons are again excluded from resonance and thermal and higher energy electrons are responsible for the
We approximate the dispersion relation in the vicinity of the second harmonic resonance by adding the $l = 2$ thermal corrections to the $l = \pm 1$ cold plasma terms, giving
\[
\frac{c^2 k^2}{\omega^2} \approx 1 - \frac{\omega_p^2}{\omega^2} - \frac{1}{\pi^{1/2}} \frac{\omega_p^2}{\omega} k^4 \rho^4 \frac{1}{32} \int_0^\infty dV \int_0^\infty \frac{2V^2 e^{-V^2}}{(\omega - 2\Omega_0(x)) [1 - (\beta/2 V^2)]} \, dV.
\]
(11)

where $\rho = v_T/\Omega, V = v/v_T$, and $\beta = v_T^2/c^2$. In order to carry out the integrations in velocity space, we transform Eq. (11) from cylindrical velocity coordinates $(V, V_z)$ to spherical coordinates $(V, \theta, \phi)$, where $V = V \cos \theta$ and $V_z = V \sin \theta$. The dispersion relation now becomes
\[
\frac{c^2 k^2}{\omega^2} = \frac{1}{\pi^{1/2}} \frac{\omega_p^2}{\omega} - \frac{1}{\pi^{1/2}} \frac{\omega_p^2}{\omega} k^4 \rho^4 \frac{1}{32} \int_0^\infty dV e^{-V^2} \int_0^\infty \frac{2V^8 \sin^4 \theta \cos^2 \theta e^{-V^2} \sin \theta d\theta}{(\omega - 2\Omega_0(x)) [1 - (\beta/2 V^2)]}.
\]
(12)

Performing the $\theta$ integration and using Eq. (8), Eq. (12) becomes
\[
\frac{c^2 k^2}{\omega^2} = \frac{1}{\pi^{1/2}} \frac{\omega_p^2}{\omega} - \frac{1}{\pi^{1/2}} \frac{\omega_p^2}{\omega} k^4 \rho^4 \frac{1}{105} \Omega^2 \beta \frac{V^8 e^{-V^2}}{(V^2/(2\beta)(x^2/L^2) + 1/\beta (\Delta \omega/\Omega_0))}.
\]
(13)

Since absorption occurs when there is a pole in the velocity integrand, again we note that for $\Delta \omega = 0$ there will be no absorption for a field maximum (the + sign) but only for a field minimum (the − sign).

Let us now treat the field minimum and maximum cases separately.

A. Field minimum, $\Omega$ mode

For the case of a field minimum, we define the quantity
\[
\eta^2 = (2/\beta L^2) [x^2 - (\Delta \omega/2\Omega_0)L^2].
\]
(14)

The integral in Eq. (13) may now be expressed in terms of the plasma dispersion function $Z$, giving
\[
\frac{c^2 k^2}{\omega^2} = \frac{1}{\pi^{1/2}} \frac{\omega_p^2}{\omega} + \frac{1}{\pi^{1/2}} \frac{\omega_p^2}{\omega} k^4 \rho^4 \frac{1}{210} \frac{\Omega^2}{\beta \Omega_0} \frac{15}{8} \eta^2 + \frac{3}{4} \eta^4 + \frac{1}{2} \eta^4 + \eta^2 Z(\eta).
\]
(15)

We now obtain Im($k$) from Eq. (15), giving
\[
\text{Im}(k) = \frac{\pi^{1/2}}{210} \frac{\omega_p^2}{\Omega_0^2} \frac{k^4 v_T^2}{\Omega_0^2} \frac{k_0 \eta^2 e^{-\eta}}{\eta},
\]
(16)

where $k_0$ is the wavenumber of the $\Omega$ mode in the cold plasma limit. The optical depth for the $\Omega$ mode crossing the second harmonic resonance in the vicinity of a field minimum is
\[
\tau = 4 \int_0^\infty \text{Im} [k(x)] \, dx.
\]  
(17)

For the case \(\Delta \omega = 0\), Eqs. (14), (16), and (17) give
\[
\tau = \left( \frac{2\pi}{105} \right)^{1/2} k_0 L \frac{\omega_p^2}{\Omega_0^2} \frac{k^2 v_T}{\epsilon} \frac{v_T}{c} I_1, 
\]
(18)
where
\[
I_1 = \int_0^\infty \eta^2 e^{-\eta^2} \, d\eta.
\]
(19)

This integral is straightforward and its value is 3, giving
\[
\tau_{\text{min}} = \frac{16}{35} \left( \frac{\pi}{2} \right)^{1/2} \frac{\omega_p^2}{\Omega_0^2} \left( 1 - \frac{\omega_p^2}{4\Omega_0^2} \right)^{1/2} \frac{v_T}{c} \frac{\Omega_0 L}{c^2}.
\]
(20)

The result for the usual linear field profile is
\[
\tau_2^2 = \pi \frac{\omega_p^2}{\Omega_0^2} \left( 1 - \frac{\omega_p^2}{4\Omega_0^2} \right)^{1/2} \frac{v_T^4}{c^4} \Omega_0 R.
\]
(21)

We note that the optical depth for a field minimum is proportional to \((v_T/c)^3\) rather than \((v_T/c)^4\) for the usual case. Thus the absorption at the second harmonic of the O mode for a field minimum will be enhanced relative to its value for a linear field gradient if
\[
v_T/c < L/5R
\]
when \(\Delta \omega = 0\). Let us now consider the situation where \(\Delta \omega \neq 0\), corresponding to cases (a) and (c) in Sec. II. For \(\omega - 2\Omega_0 > 0\), cyclotron resonance can only occur when
\[
x^2 > (\Delta \omega/2\Omega_0)^2 L^2 \equiv x_c^2.
\]
(23)

This is illustrated in Fig. 1. The optical depth given by Eqs. (14), (16), and (17) becomes
\[
\tau = \frac{2\pi}{105} k_0 L \frac{\omega_p^2}{\Omega_0^2} \frac{k^2 v_T}{\epsilon} \frac{v_T}{c} I_1(\alpha),
\]
(24)
where
\[
I_1(\alpha) = \alpha^4 e^\alpha \int_0^\infty (t^2 - 1)^{7/2} e^{-\alpha t^2} \, dt
\]
(25)
and
\[
\alpha = \Delta \omega/\beta \Omega_0.
\]
(26)

In Fig. 4, \(I_1\) is plotted as a function of \(\alpha\) in curve (a), which shows that the optical depth remains within 50% of its value for \(\Delta \omega = 0\) provided \(\Delta \omega/2\Omega_0 \leq 5v_T/c^2\). Now consider \(\omega - 2\Omega_0 < 0\); as noted in Sec. II, this condition excludes low-energy electrons from resonance. In particular, for a certain value of \(\Delta \omega\), thermal electrons will resonate with the wave in the region where the rate of variation of the magnetic field is least. The optical depth given by Eqs. (14), (16), and (17) is now
\[
\tau = \frac{2\pi}{105} k_0 L \frac{\omega_p^2}{\Omega_0^2} \frac{k^2 v_T}{\epsilon} \frac{v_T}{c} I_2(\alpha),
\]
(27)
where \(\alpha\), defined by Eq. (26), is negative and
\[
I_2(\alpha) = \alpha^4 e^\alpha \int_0^\infty (t^2 + 1)^{7/2} e^{-\alpha t^2} \, dt.
\]
(28)

In Fig. 4, \(I_2\) is plotted as a function of \(-\alpha\) in curve (b). A maximum occurs for \(-\alpha \approx 2.4\), where the optical depth is increased by 50% compared with its value when \(\Delta \omega = 0\), given by Eq. (20). The optical depth falls to less than 50% of the value for \(\Delta \omega = 0\) only when \(\alpha < -6\). The region of strong absorption for the O-mode second harmonic is given approximately by
\[
\frac{3v_T^2}{c^2} \leq \Delta \omega \leq \frac{5v_T^2}{c^2}.
\]
(29)

It is interesting that for this case the absorption can be stronger than for the case \(\Delta \omega = 0\).

**B. Field maximum, O mode**

Let us now consider the absorption in the vicinity of a field maximum. We define a quantity \(\xi\), corresponding to \(\eta\) introduced in the field minimum case,
\[
\xi^2 = \frac{2}{\beta L^2} \left[ (\Delta \omega/2\Omega_0)^2 L^2 - x^2 \right].
\]
(30)

We have already noted that for \(\Delta \omega > 0\), there is no absorption in the vicinity of a field maximum, since the whole region is a low-field zone where no electrons can come into cyclotron resonance with the wave. However, when \(\Delta \omega < 0\), electrons can resonate with the wave provided
\[
x^2 < (\Delta \omega/2\Omega_0)^2 L^2 \equiv x_c^2.
\]
(31)

Again using Eqs. (14), (16), and (17), the optical depth becomes
\[
\tau = \frac{2\pi}{105} k_0 L \frac{\omega_p^2}{\Omega_0^2} \frac{k^2 v_T}{\epsilon} \frac{v_T}{c} I_3(\alpha),
\]
(32)
where \(\alpha\), defined by Eq. (26), is negative and
\[
I_3(\alpha) = \alpha^4 e^\alpha \int_0^\infty (1 - t^2)^{7/2} e^{-\alpha t^2} \, dt.
\]
(33)

Curve (c) of Fig. 4 plots \(I_3\) as a function of \(-\alpha\), showing a maximum at \(-\alpha \approx 5.4\). In this case, the optical depth given by Eq. (32) is increased by a little over 30% compared with
its value for $\Delta \omega = 0$ at a field minimum. It can be seen from Fig. 4 that the region of strong absorption for the O-mode second harmonic for the case of a field maximum is given approximately by

$$-\frac{9v_T^2}{c^2} \leq \frac{\Delta \omega}{2\Omega_0} \leq -\frac{v_T^2}{c^2}.$$  \hspace{1cm} (34)

C. X mode

We now turn to an analysis of the X mode at the third harmonic. The dispersion relation for the X mode propagating perpendicular to the magnetic field can be written

$$D_{xx}D_{yy} - D_{xy}D_{yx} = 0,$$  \hspace{1cm} (35)

where

$$D_{xx} = 1 - 2\pi \frac{\omega_p^2}{\omega} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dv_{||}$$
$$\times \int_{0}^{\infty} \frac{2v_{||} dv_{\perp}}{\omega - \omega} I_l^2 \Omega_0^2 \frac{k v_{||}}{k} J_l^2 \left( \frac{k v_{||}}{\Omega} \right) \frac{df_0}{dv_{\perp}^2},$$  \hspace{1cm} (36)

$$D_{yy} = 1 - 2\pi \frac{\omega_p^2}{\omega} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dv_{\perp}$$
$$\times \int_{0}^{\infty} \frac{2v_{\perp} dv_{||}}{\omega - \omega} I_l^2 \Omega_0^2 \frac{k v_{||}}{k} J_l^2 \left( \frac{k v_{||}}{\Omega} \right) \frac{df_0}{dv_{\perp}^2},$$  \hspace{1cm} (37)

$$D_{xy} = -2\pi i \frac{\omega_p^2}{\omega} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} dv_{||}$$
$$\times \int_{0}^{\infty} \frac{2v_{||} dv_{\perp}}{\omega - \omega} I_l \Omega_0 v_{||} J_l \left( \frac{k v_{||}}{\Omega} \right) J_l \left( \frac{k v_{||}}{\Omega} \right) \frac{df_0}{dv_{\perp}^2},$$  \hspace{1cm} (38)

$$D_{yx} = -D_{xy}.$$  \hspace{1cm} (39)

Again we assume a small electron Larmor radius, so that $k v_{||} \ll \Omega$, and expand the Bessel function for $l = 3$:

$$J_3 \left( \frac{k v_{||}}{\Omega} \right) \approx k^3 v_{||}^3 / 48 \Omega^3.$$  \hspace{1cm} (40)

The dispersion relation is simplified, as before, by including only the cold, $l = \pm 1$ terms, and the thermal corrections from the resonant ($l = 3$) term. We obtain, after some algebra,

$$\frac{c^2 k^2}{\omega^2} = \frac{\omega (\omega + \Omega) - \omega_p^2}{(\omega^2 - \omega_p^2 - \Omega^2)\omega^2}$$
$$+ \left[ \frac{1 - \omega_p^2}{(\omega^2 - \omega_p^2)} \right] D^{(3)}_{xx} + D^{(3)}_{yy} - \frac{c^2 k^2}{\omega^2} D^{(3)}_{xx}$$
$$- 2i \frac{\Omega}{\omega (\omega^2 - \Omega^2)} \left[ D^{(3)}_{xy} + D^{(3)}_{yx} \right]$$
$$\times \left( \frac{\omega^2 - \Omega^2}{\omega^2 - \omega_p^2 - \Omega^2} \right).$$  \hspace{1cm} (41)

The quantities $D^{(3)}_{ij}$ are as follows:

$$D^{(3)}_{xx} = -\frac{k^4 \rho^4 \omega_p^2}{64} \frac{1}{\pi \omega^2} \int_{-\infty}^{\infty} dV_{\parallel}$$
$$\times \left[ \int_{0}^{\infty} \frac{v^2 e^{v^2} - v^2 e^{-v^2}}{(\omega - 3\Omega_0(x) [1 - (\beta/2) V^2])} dV_{\perp} \right],$$  \hspace{1cm} (42)

$$D^{(3)}_{yy} = D^{(3)}_{zz},$$  \hspace{1cm} (43)

$$D^{(3)}_{xy} = iD^{(3)}_{yx}.$$  \hspace{1cm} (44)

The dispersion relation then becomes

$$\frac{c^2 k^2}{\omega^2} = \frac{\omega (\omega + \Omega) - \omega_p^2}{(\omega^2 - \omega_p^2 - \Omega^2)\omega^2}$$
$$+ \left[ \frac{1 - \omega_p^2}{\omega (\omega + \Omega)} \right] \frac{c^2 k^2}{\omega^2}$$
$$\times \left[ \frac{\omega^2 - \Omega^2}{\omega^2 - \omega_p^2 - \Omega^2} \right] D^{(3)}_{xx}.$$  \hspace{1cm} (45)

Making use of Eq. (5), we define the quantity

$$\mu^2 = (2/\beta \xi^2) [x^2 - (\Delta \omega/3\Omega_0) L^2],$$  \hspace{1cm} (46)

which corresponds to the case of a field minimum; $\Delta \omega$ is now given by $\Delta \omega = \omega - 3\Omega_0$. Introducing Eq. (46) into Eq. (41) and transforming to spherical velocity coordinates, the resonant integral is found to be the same as that in Eq. (13). Employing the results of the previous analysis, we solve Eq. (45) perturbatively to obtain

$$\text{Im}(k) = \frac{27}{140} \frac{\pi^{1/2} v_T^3}{c^2}$$
$$\frac{\omega_p^2}{\Omega_0} \frac{(1 - \omega_p^2/12 \Omega_0^2)^{3/2}}{(1 - \omega_p^2/8 \Omega_0^2)^{3/2}}$$
$$\times \left( \frac{1 - \omega_p^2}{6 \Omega_0^2} \right)^{3/2} \frac{\Omega_0 L}{c} \mu^2 e^{-\mu^2},$$  \hspace{1cm} (47)

where we have put $\omega = 3\Omega_0$ in nonresonant terms. We use Eq. (17) for the case $\Delta \omega = 0$ to obtain the optical depth for the X mode crossing the third harmonic resonance in the vicinity of a field minimum:

$$\tau_{\text{min}} = \frac{81}{35} \left( \frac{\pi}{2} \right)^{1/2} \frac{v_T^3}{c^2}$$
$$\frac{\omega_p^2}{\Omega_0} \frac{(1 - \omega_p^2/12 \Omega_0^2)^{3/2}}{(1 - \omega_p^2/8 \Omega_0^2)^{3/2}}$$
$$\times \left( \frac{1 - \omega_p^2}{6 \Omega_0^2} \right)^{3/2} \frac{\Omega_0 L}{c}.$$  \hspace{1cm} (48)

In terms of the wavenumber $k_x$ of the X mode in the cold plasma limit, given by

$$k_x^2 = (12 \Omega_0^2 - \omega_p^2) (6 \Omega_0^2 - \omega_p^2)/c^2 (8 \Omega_0^2 - \omega_p^2),$$  \hspace{1cm} (49)

the optical depth can be written

$$\tau_{x} = \left( \frac{\pi}{2} \right)^{1/2} k_x \rho \frac{\omega_p^2}{\Omega_0^2} \left( \frac{12 - \omega_p^2/\Omega_0^2}{8 - \omega_p^2/\Omega_0^2} \right)^2 \frac{\Omega_0 L}{c}.$$  \hspace{1cm} (50)

The optical depth for the X mode crossing the third harmonic resonance in a linear gradient is

$$\tau_x = \frac{\pi}{12} \frac{\omega_p^2}{\Omega_0^2} k_x \rho \frac{v_T}{c} \left( \frac{12 - \omega_p^2/\Omega_0^2}{8 - \omega_p^2/\Omega_0^2} \right)^2 \Omega_0 R.$$  \hspace{1cm} (51)

It follows that the condition for the field minimum to produce an enhancement in the absorption is identical to Eq.
We also note that the ratio of the optical depths for the third harmonic X mode and the second harmonic O mode in the vicinity of a field minimum is

\[ \frac{\tau_{X_{\text{min}}}(l = 3)}{\tau_{O_{\text{min}}}(l = 2)} = \frac{2}{3} \frac{(12\Omega_0^2 - \omega_p^2)^{7/2}}{(8\Omega_0^2 - \omega_p^2)^{7/2}} \frac{(6\Omega_0^2 - \omega_p^2)^{3/2}}{(4\Omega_0^2 - \omega_p^2)^{3/2}}. \]  

(52)

Evidently the third harmonic X mode is the more strongly absorbing. Since the resonant integral for the third harmonic X mode is the same as that for the second harmonic O mode, all the previous conclusions apply.

IV. EQUILIBRIUM MAGNETIC FIELD EXTREMUM

We now consider, very briefly, the conditions for the occurrence of an extremum in the equilibrium magnetic field in a tokamak. This can occur along the equatorial midplane of a tokamak in a high-beta equilibrium. Figure 5(a) gives an example for parameters typical of the COMPASS tokamak at a volume average beta value of 5%. This figure is a computer simulation of the COMPASS tokamak obtained from a Grad–Shafranov equilibrium code and shows the variation of $|B|$ with major radius $R$ with a neighboring shallow maximum and minimum. The toroidal current profile $J_{\phi}(R)$ is shown in Fig. 5(b) where the steep current gradient accounts for the rather abrupt return from the shallow magnetic field profile in the vicinity of the extremum to the usual linear profile. An approximate estimate of the scale length $L$ in the extremum region gives $L \approx 2m$, which is to be compared with the major radius of the COMPASS device, $R = 0.55$ m. Thus in this case $L/R \approx 4$, and the enhancement factor of $\tau^3$ relative to that obtained for a linear $x/R$ variation of $|B|$ is of order 20 for an electron temperature of 1 keV.

In a low-beta tokamak equilibrium, $|B|$ is typically a monotonic function of $R$ along the equatorial midplane, but along vertical chords (i.e., $\theta = \pi/2$) $|B|$ can have maxima or minima. In the large aspect ratio ($\epsilon = a/R \ll 1$), circular cross-section approximation, for example, $B(r)$ is given by

\[ B \approx B_0[1 + g(r) + r^2/2R^2 q^2], \]  

(53) along the vertical chord through the magnetic axis. In this expression, the $r^2/q^2$ term arises from the poloidal magnetic field, while the diamagnetic (or paramagnetic) modification to the toroidal field is contained in $g(r)$, which is determined by the pressure balance equation

\[ \frac{d}{dr} \left( g + \frac{p(r)}{B_0^2} \right) + \frac{1}{Rq^2} \frac{d}{dr} \left( \frac{r^2}{q} \right) = 0. \]  

(54)

Here the $p(r)$ and $J_{\phi}(r)$ profiles are given by

\[ p = p_0(1 - r^2/a^2)^n, \quad J_{\phi} = J_0(1 - r^2/a^2)^n, \]  

(55) in which case

\[ \frac{1}{L^2} \equiv \frac{1}{2B} \frac{d^2B}{dr^2} | \bigg|_0 = \frac{1}{2a^2} \left( a\beta_0 - \frac{a^2}{Rq_0^2} \right), \]  

(56)

with diamagnetic behavior when $\beta_0$ exceeds the critical value $\beta_c = a^2/\alpha R^2 q_0^2$, and paramagnetic behavior otherwise. For the COMPASS parameters discussed above, we again obtain $L/R = 4$ and a minimum in $B(r)$ for this case.

V. CONCLUSIONS

We have calculated the optical depths for the second harmonic O mode and the third harmonic X mode in the vicinity of an extremum of the equilibrium magnetic field. The dependence of these optical depths on the temperature was found to be $(T/e/mc^2)^{3/2}$, instead of $(T/e/mc^2)^2$ for the usual monotonic field variation. Since the scale length $L$ of the magnetic field variation in the case of a field extremum is larger than the scale length $R$ of monotonic variation, the absorption in the region of a field extremum can be enhanced. The criterion for this was shown to be $\nu_f/c < L/5R$. This enhancement will occur for both a field minimum and a field maximum.

In order to utilize this effect, the source frequency would require fairly precise tuning to the resonance region. For a field minimum, the source frequency should be tuned to the minimum position, and enhanced absorption occurs for $-3\nu_f^2/c^2 < \Delta \omega/\omega < 5\nu_f^2/c^2$. For a field maximum, the frequency should be tuned to $\Delta \omega/\omega = -3\nu_f^2/c^2$, and enhanced absorption occurs for $-9\nu_f^2/c^2 < \Delta \omega/\omega < -\nu_f^2/c^2$. For both a field minimum and a field maximum, the peak in the absorption does not occur when the frequency is tuned to

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the extremum position. Instead, the peaks occur when \( \Delta \omega = \omega - \Omega_0 < 0 \). This excludes low-energy electrons from resonance with the wave for the field minimum case. In the field maximum case, low-energy electrons are also excluded from resonance, but only in the immediate neighborhood of the position of the maximum.

Field extrema occur both for finite-beta plasmas and for low-beta plasmas along a vertical chord. For a ray following a vertical chord, refraction would also be significant; it would cause the ray to sample the usual toroidal variation in addition to the more gradual change in the vertical direction. In this case, since the variation of the magnetic field in the vertical direction is so small, the variation in the density and temperature should also be included. Rays that are refracted to the low-field side would quickly move out of resonance whereas those refracted to the high-field side would stay in resonance longer. This case requires a two-dimensional treatment.

In order to quantify the enhancement in the absorption, we require an estimate of the scale length \( L \). This was obtained in Sec. IV, where we estimated a value \( L/R \approx 4 \) from a COMPASS equilibrium code. Using the COMPASS major radius \( R = 0.55 \) m, choosing a magnetic field of 1 T, and assuming \( \omega_p^2/\Omega_0^2 = 4 \), we obtain an optical depth for the X mode at third harmonic resonance \( \tau_X^2 \approx 2.3 \) for \( T_e \approx 1 \) keV. Similarly, for the O mode at second harmonic resonance, again choosing a field of 1 T with \( \omega_p^2/\Omega_0^2 = 2 \), the optical depth \( \tau_O^2 \approx 1 \) for \( T_e = 4 \) keV. Thus both these modes, which give access to higher densities, become strongly absorbing in the vicinity of a field extremum.

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