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Toroidal ion temperature gradient-driven weak turbulence

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In this paper, a theory of toroidal ion temperature gradient-driven weak turbulence near the threshold is presented. The model considers gyrokinetic ions and adiabatic electrons in toroidal geometry. The linear theory considers modes with \( k_{\parallel} v_{\parallel} \sim O(1) \) (generally not considered in toroidal theories), giving a toroidal threshold regime dominated by transit resonances (as opposed to the more usually considered drift resonances), and with a stability threshold of

\[
\eta_{\text{tor}}^{\text{thr}} \approx 1 + 2 \epsilon \left( 2/\tau + 1/(1 + 3) \right). \tag{1}
\]

It is shown that when \( 0 < \eta_{\parallel} - \eta_{\text{tor}}^{\text{thr}} < 2 \sqrt{1 + 1/\tau} \times L_{\parallel}/\sqrt{RL_{\parallel}} \) then \( 0 < \gamma < \omega_{\parallel} \), and a weak turbulence expansion can be used to treat the nonlinearity. The instability is saturated via nonlinear ion resonance, and it is shown that this nonlinear process transfers energy directly from the waves to the ion distribution function, and does not conserve wave energy. The saturated spectrum is calculated, and the resulting ion thermal conductivity is found to be

\[
\chi_{\parallel} = \sqrt{1 + 1/\tau} \left( \frac{\eta_{\parallel} - \eta_{\text{tor}}^{\text{thr}}}{\eta_{\parallel}} \right) \left( L_{\parallel}/R \right)^{3/2} \rho_{i} \Omega_{i}, \tag{2}
\]

which is smaller than typical mixing length estimates by a factor of about \( L_{\parallel}/R \), but in the range of tokamak observations. The diffusive nature of the transport (requiring a short radial step size) is reconciled with the broad radial extent of the toroidally coupled linear modes by postulating that it is the nonlinear beat wave between linear modes (not the radial width of a single linear mode) that determines the step length appropriate for transport.

I. INTRODUCTION

The theory of ion temperature gradient-driven turbulence ("\( \eta_{\parallel} \) turbulence") is able to explain many qualitative features of tokamak confinement,\,1-3 but quantitatively the theory and local experimental measurements can sometimes differ by an order of magnitude.\,4-7 However, the available theoretical predictions are generally based on rather idealized models, and several studies have shown that there can be significant quantitative corrections (orders of magnitude) coming from effects which are often neglected in the interest of analytical tractability. No theory to date has combined all such effects and produced a prediction of the transport, and so it can be argued that disagreements between existing theories and experiment are inconclusive. A more complete theory is required, and the present work is part of an effort to provide this.

The paradigm model of the \( \eta_{\parallel} \) instability has generally been the slab fluid model. This mode is basically an ion sound wave parallel to the magnetic field, which accesses the free energy source of the radial ion temperature gradient.8-10 Nonlinearly, this mode evolves to the strong turbulence regime, and gives rise to an ion thermal conductivity comparable to the mixing length level,\,11-13 \( \chi_{\parallel} \sim \gamma/\kappa_{\parallel}^{2} \). Although this slab fluid model has been useful for exploring the qualitative behavior of the mode, the quantitative accuracy is uncertain at best. Threshold and toroidal effects can cause significant modifications to the predictions.

Near the threshold of the instability (often the regime of experimental interest), several features of the basic fluid \( \eta_{\parallel} \) mode are altered. Linear ion resonances become important (occurring in the slab limit for particles with \( \omega = k_{\parallel} v_{\parallel} \)), and damp the mode,\,1,11 so that the mode becomes fully stabilized when the density gradient becomes steeper than the ion temperature gradient (i.e., when \( \eta_{\parallel} \equiv d \ln T_{i}/d \ln n_{i} \lesssim 1 \)). Nonlinear ion resonances (also known as nonlinear Landau damping or ion Compton scattering) accompany these linear resonances,\,14 and replace the fluid saturation mechanism (multiple wave resonance). Additionally, when the growth rate is smaller than the frequency, \( \gamma < \omega_{\parallel} \), the commonly used estimates from the strong turbulence ordering (such as the mixing length estimate) break down, and a wave kinetic formulation\,15 is more appropriate. This threshold regime has been explored in Ref. 14 in slab geometry. Although the threshold regime covers only a certain range of \( \eta_{\parallel} \) near the threshold, the results give insight into the behavior of nonlinear Landau damping, which can be expected to be important at higher values of \( \eta_{\parallel} \) for modes of higher radial wave number (often denoted by \( l \)).

Toroidicity enters through the magnetic drift velocity, \( v_{\parallel} \), and also introduces several major changes in the basic picture. One change is the introduction of a new destabilization mechanism, coming from a pressure gradient in the presence of unfavorable toroidal curvature (analogous to the classical Rayleigh–Taylor instability), which competes with the sound wave destabilization mechanism of the basic instability. A second change is the introduction of toroidal coupling, which linearly joins modes with the same toroidal mode number \( n_{t} \), and tends to make the modes a good deal broader radially. Toroidally coupled modes are generally described by a two-dimensional partial differential equation (PDE), which can be rendered tractable via introduction of the ballooning formalism.\,16 Finally, the introduction of \( v_{\parallel} \) complicates the ion resonance condition, which becomes

\[
\omega - k_{\parallel} v_{\parallel} - k_{\perp} v_{\parallel} = 0, \tag{3}
\]

and thus acquires a quadratic de-
dependence on both $v_1$ and $v_1$.

The net effect of toroidicity on the $\eta_i$ mode is not well established, since the literature contains a diversity of views. One such point is the relation between the toroidal (bad curvature) and slab (sound wave) destabilization mechanisms. Horton et al.\textsuperscript{17} find that the character of the mode depends on the macroscopic geometry, namely, that when $2L_\perp < R$ the modes are slablike, changing over to a ballooning-like structure in the opposite limit. Guo et al.\textsuperscript{18} find that the mode changes character depending on the regime in k space, such that toroidal destabilization dominates when $k_0 \rho_0 \ll L_\perp/R$, and becomes negligible in the opposite limit.

(Although their study is nominally restricted to the regime $L_\gamma \to \infty$, it appears that similar results can be obtained when this condition is relaxed.\textsuperscript{19}) Other studies\textsuperscript{20} use the term slab and toroidal “branches,” implying a picture of two coexisting instabilities. A clearer picture of the relation between the slab and toroidal destabilization mechanisms is needed.

A second point of uncertainty is whether parallel ($\omega \approx k_1 v_1$) or drift ($\omega \approx k_1 v_\parallel$) resonances (or neither) are important in the vicinity of the threshold. Various authors retain either the former\textsuperscript{21} or the latter\textsuperscript{19,22} and neglect the other, or neglect both,\textsuperscript{23} but never with any apparent assessment of the relative magnitude. Although any of these techniques apparently produces roughly the same linear stability threshold, the transport (of concern in the present study) is more sensitive to the details of the dynamics, and so the resonance issue must be addressed. A third point of uncertainty is the determination of the radial width $\Delta r$ of the toroidally coupled modes. The studies of Hastie et al.\textsuperscript{24} and Connor and Taylor\textsuperscript{25} find that the radial envelope of the linear modes is determined by terms that vary as $d^2 \omega_i / dr^2$. Since such terms are generally ignored in $\eta_i$ mode theory, this implies $\Delta r = \infty$ to the order generally considered. Guo et al.\textsuperscript{18} argue that when toroidal destabilization is subdominant, the modes are not toroidally coupled and the separation of rational surfaces can be used, $\Delta r \sim 1/k_\parallel \delta$. Biglari et al.\textsuperscript{26} use the fact that the ballooning coordinate behaves like the Fourier conjugate to the radial coordinate to argue that $\Delta r = 1/k_\parallel \delta \Delta_\perp$, where $\Delta_\perp$ is the parallel width of the quasimode in ballooning space. Given this diversity of viewpoints, it is important to resolve the issue of what determines $\Delta r$, since the choice makes a significant difference on the scalings, magnitude, and underlying nature (diffusive or convective) of the resulting transport. Finally, Hirose and Ishihara\textsuperscript{27} claim that toroidal effects completely stabilize the mode, which stands in contradiction to most other studies. In summary, the present understanding of the toroidal $\eta_i$ mode can be characterized as having a number of diverging views with few studies attempting to bridge the gaps.

In this paper, we present a theory of weak toroidal $\eta_i$ turbulence near the threshold of instability, thus combining the effects mentioned above (and in the process, new results emerge regarding both). The geometrical model consists of concentric circular flux surfaces, with radial gradients of density and ion temperature. Fluctuations are electrostatic, and described by the nonlinear ion gyrokineheuristic equation,\textsuperscript{27} adiabatic electrons and quasineutrality. The ballooning formalism\textsuperscript{16} is used to treat the toroidal mode structure. This complicated set of equations is rendered tractable by treating $\gamma/\omega_i < 1$ as a small parameter (where $\omega_i$ and $\gamma$ are the linear frequency and linear growth rate, respectively). With this ordering, the lowest-order equation describes $\omega_i$ and the linear mode structure, and at first order is the wave kinetic equation, describing linear growth and nonlinear saturation of the modes. The condition $\gamma/\omega_i < 1$ defines the validity regime for the theory, which may be termed the “threshold regime.” In the linear theory, transit resonances and sound wave dynamics (often neglected \textit{a priori}) are retained via the “curvature approximation,” and it is found that they can give the dominant contribution to the growth rate near the threshold (rather than the more usually considered drift resonances and curvature effects). A new method of solving the parallel mode equation is introduced, whereby asymptotically vanishing solutions are constructed by matching to the Bloch functions (also known as Floquet solutions). This produces different results from the more usual techniques.

In the nonlinear theory, the wave kinetic equation is solved to yield the quasimode spectrum in both $k_\parallel$ and $k_\perp$. The presence of different $k_\parallel$ in the spectrum is particularly important here, since interaction of modes with different $k_\parallel$ produces fluctuations with a small radial correlation length, consistent with diffusive radial transport (the linear modes have a radial correlation length that is essentially infinite). This point is discussed further in Sec. V.

The main results of this work are as follows.

(1) For $\eta_i$ in the range $0 < \eta_i - \eta_{th,0} < 2/1 + 1/2$ the toroidal $\eta_i$ mode is in the “threshold regime,” characterized by weak turbulence and the prominence of ion resonances in the basic linear and nonlinear dynamics.

(2) The toroidal turbulence is found for which sound wave destabilization (not unfavorable curvature) is the dominant drive. As a result, unstable linear modes can be located in regions of both good and bad curvature.

(3) The toroidal linear stability threshold is found to be

$$\eta_{th,0} = 1 + 2\varepsilon_n [2/r + 1/(1 + 3)],$$

where $\tau = T_e/T_i$, $\delta = d \ln q/d \ln r$, and $\varepsilon_n = L_n/R$.

(4) The mode saturates by nonlinear parallel Landau damping, whereby an ion resonates with the beat wave between two modes. Results indicate that this process transfers energy from the mode with lower $k_\parallel$ to the ions, so that turbulence energy is not conserved nonlinearly.

(5) The quasimode spectrum in the toroidal threshold regime is found to be

$$\left| \delta \dot{\phi}_k \right|^2 = \frac{1}{(2\pi)^2} \int \frac{1}{\ln \delta^{\parallel}/\delta^{\perp}} \left| \frac{\eta_i - \eta_{th,0}(k)}{R^2 k_\perp^2} \right|$$

and $\dot{\phi}_k = 0$ in regions of k space, where $\eta_i < \eta_{th,0}$. (The logarithmic factor in the denominator is regarded to be complicated, but with weak scalings and of order unity; here, $\xi = \omega_i/\sqrt{2} k_\parallel v_{th,i}$ and $\theta_0 = k_\parallel \delta$, where $v_{th,i}$ is the ion thermal velocity.) The spectrum is relatively flat in $\dot{\theta}_0$ (the index of the center of the quasimode), due to the growth rate having weaker dependence on $\dot{\theta}_0$ than on $\dot{\theta}_0$.
(6) The ion thermal conductivity in the threshold regime is

$$Y_i = \sqrt{1 + \frac{1}{\tau} \left( \frac{\eta_i - \eta_{th}}{\eta_i} \right) \frac{L_T}{R \sqrt{r^2 + \rho_i^2} \Omega_i}},$$

which is less than the fluid regime value\textsuperscript{29} by an approximate factor of $L_T/R$. It is found that the estimate $\chi_i = \xi^2 \gamma/k^2$ is more appropriate than the mixing length estimate for the present case (where $\xi$ is defined in item (5) above).

The remainder of this paper is organized as follows. In Sec. II we outline the basic model and equations. In Sec. III A, the linear mode equation is derived, and in Sec. III B this is solved to yield the frequency, growth rate, and basic mode structure in the neighborhood of the threshold. Section IV contains the nonlinear theory, including a weak turbulence derivation of the damping from nonlinear ion Landau resonances, and a calculation of the saturated spectrum. Section V contains a derivation of the ion thermal transport associated with the saturated spectrum, and Sec. VI is a summary and conclusions.

II. MODEL

The geometry of the toroidal $\eta_i$ mode is described here by circular concentric toroidal flux surfaces, with radial gradients of density and ion temperature. The coordinate system is defined by $r$ (radial), $\theta$ (poloidal), and $\varphi$ (toroidal) directions, oriented such that \( \hat{r} \times \hat{\varphi} = 1 \). The magnetic field is given by $B = |B| \hat{B}$, where $|B| = B_0 \left[ 1 - (r/R) \cos \theta \right]$ and $\hat{B} = \hat{\varphi} + (B_0/B_e) \hat{\theta}$.

With such a geometry, the toroidally coupled modes can be described by the ballooning representation,\textsuperscript{16} whereby

$$\hat{\phi}(x,t) = \int d\chi \sum_{n} \sum_{m} \exp \left[ i(nq - m\theta + k_r r + k_r^* r') \right] \hat{\phi}_k(r,\theta,t),$$

where the quasimode $\hat{\phi}_k$ is required to vanish asymptotically in $\theta$. Equation (1) represents $\hat{\phi}(x)$ as a sum of Fourier modes, indexed with $n$, $m$, and $k_r$. It is also possible, by summing the modes of different $m$, to represent $\hat{\phi}(x)$ as a sum of quasimodes, which have a structure that is extended radially and twists with the sheared field lines:

$$\hat{\phi}(x,t) = 2\pi \int d\chi \sum_{n} \sum_{m} \exp \left[ i(nq - nq \theta + 2\pi p) \right] \hat{\phi}_k(r,\theta + 2\pi p_\theta,t).$$

Equations (1) and (2) are equivalent linearly, but in a random phase nonlinear theory it must be decided whether it is the Fourier modes on the quasimodes that acquire the random phase factor. For the present situation, note that the modes are in the weak turbulence regime, for which the linear and nonlinear mode structures are not appreciably different. Thus the turbulence must be represented as a sum of randomly phased quasimodes, since these are the correct linear modes, and a random phase is assigned to each $n$ and $k_r$ (although periodicity in $\theta$ dictates that modes with only $\rho$ different have the same phase). It must be emphasized that there is no radial envelope for the quasimodes unless explicitly determined\textsuperscript{24,25} from terms that vary as $d^2 \omega_\omega/d\rho^2$, which is not attempted here, nor apparently in any other analysis of the toroidal $\eta_i$ mode. In the present model, a finite radial correlation length arises nonlinearly through the coexistence of modes with different $k_r$, a point developed further in Sec. V.

Fluctuations are electrostatic, and described by the nonadiabatic part of the ion distribution function, $f(x,v,t)$, and the electrostatic potential $\phi(x,t)$. It is assumed that fluctuations have perpendicular length scales much smaller than those of the equilibrium gradients, and time scales in the regime $v_i - \omega/k \ll v_e$, so that ions are resonant and electrons are adiabatic. With these assumptions, the fields evolve by the nonlinear ion gyrokinetic equation written in the ballooning formalism\textsuperscript{27}

$$(\frac{\partial}{\partial t} + \nabla_\parallel \cdot + i\omega_{Di}) \hat{\phi}_k(\hat{\theta},\mathbf{v},t) - F_M J_0 (k_r v_i) \left( \frac{\partial}{\partial t} + i\omega_{\omega_j} \right) \times \hat{\phi}_k(\hat{\theta},t) = 2\pi \sum_{n'} \sum_{n''} \int dk' \hat{3} k' k'' \times \left[ \theta_v - \theta_v^{e_\theta} + 2\pi (p' - p'') J_0 (k'_v v_i) \right] \times \hat{\phi}^{k}_r(\hat{\theta},2\pi p''/v_i)$$

(3)

with $k'' = k_r - k_r' - 2\pi p' k'' - 2\pi p'' k''$ required for radial three wave matching), and the quasineutrality equation with adiabatic electrons:

$$\left( 1 + \frac{1}{\tau} \right) \hat{\phi}_e(\hat{\theta},t) = \int d^3 v J_0 (k_r v_i) \hat{\phi}_k(\hat{\theta},\mathbf{v},t).$$

Here $\hat{h}$ is the nonadiabatic part of the ion distribution function, $\phi$ is the electrostatic potential normalized to $e/T_i$, $\hat{\theta}$ is the parallel coordinate in ballooning space, $k_{\theta} = nq/r$ is the poloidal wave vector, $k_r$ is the absolute radial wave vector that indexes the quasimode, $\theta_v = k_r/3k_{\theta}$, $k_r(\hat{\theta}) = k_{\theta}3(\hat{\theta} - \theta_{\theta})$ is the local radial wave vector, $k_r' = k_r$ is the square of the local perpendicular wave vector, $F_M$ is a local Maxwellian with radial $T_v$, and $n$ gradients, $J_0$ is a Bessel function,

$$\omega_{Di} = -e_{\omega} \left[ \cos \hat{\theta} \cdot \hat{3} (\hat{\theta} - \theta_v) \sin \hat{\theta} \right] (v_i^2 + v_e^2/2),$$

$$\omega_{\omega_j} = e_{\omega} \left[ 1 - \eta_i (3 - \nu^2) / 2 \right],$$

$$\eta_i = L_n / L_T, L_n^{-1} = -d \ln n / d r, L_T^{-1} = -d \ln T / d r, \delta = d \ln q / d \ln r, q is the safety factor, \tau = T_i / T_v, e_{\theta} = L_n / L_T, e_{\theta} = L_n / L_T, R is the major radius, v_i^2 = T_i / n_i, \Omega_i = eB / m_{<} c_\rho, v_i = v_i / \Omega_i, and distance and time have been undimensionalized to $\rho_i$ and $1/\Omega_i$.

III. LINEAR THEORY

Although the linear theory of the toroidal $\eta_i$ mode has been examined previously by several authors,\textsuperscript{19,21-23} these studies have been concerned only with the linear stability threshold. Here, a more detailed linear theory must be undertaken, in order to obtain the information necessary to resolve nonlinear saturation and transport. One issue is whether drift or parallel resonances are more important in this regime. The uncertainty arises because the velocity

space integral, Eq. (5) below, is difficult to perform while keeping both resonances, and so the general approach has been to retain one and expand the other, making assessment of the relative contributions difficult. It appears that whichever expansion is performed, the linear threshold is insensitive (crudely speaking) and varies as $\eta_{i-1}^{-1} + O(L_n/R)$, but the nonlinear resonances considered in Sec. IV may be more sensitive to the resonance type. Other issues that need resolution in the linear theory are the value of $k_\parallel$ (since this determines the radial correlation length for transport), the parallel mode structure (which determines $k_\parallel$), and the ordering of $\omega_\parallel$, $\gamma$, $k_\parallel$, and $k_\theta$ for the unstable modes.

A. Linear mode equation

The parallel structure of the toroidal $\eta_i$ modes may be described by an integral equation in $\hat{q}_{\parallel}^{10}$ but such a formulation tends to be analytically intractable and is abandoned here. Instead the parallel dynamics are described semi-loccally, meaning that $\nabla_{\parallel}$ is used interchangeably with $i k_\parallel = i k_{\parallel}/qR$, which is then treated as a number and not as an operator (consistent with a WKB description of the mode structure). (Here $k_{\parallel}$ represents the dimensionless wave number in the ballooning angle, and is not to be confused with the poloidal wave number $k_\phi = n q/r$.)

Such a description applies when the solution has a shorter scale length than the coefficients in the equation, which can be verified a posteriori here. With this, the linearized version of Eq. (3) can be solved by Fourier transforming in $l$ and substituting $\hat{h}_k$ into Eq. (4) producing

$$\left(1 + \frac{1}{\tau}\right) \hat{h}_k = \int d^3v \frac{\omega - \omega_k}{\omega - k_{\parallel} v_{\parallel} - \omega_{Di}} J_0^2(k_{\|} v_{\parallel}) F_{kl} \hat{\phi}_k,$$

(5)

The usual way to examine the toroidal $\eta_i$ threshold is (a) to expand the denominator in Eq. (5) for $k_{\parallel} v_{\parallel} < \omega_{Di}$ in order to perform the velocity integral, and then (b) to assume that the mode is localized to the bad curvature region, which requires that $k_{\parallel} \sim 1/qR$. However, it appears that these two steps are in conflict (at least from the viewpoint of the local approach), since for both inequalities to be possible the condition $[\cos \hat{q} + \sin (\hat{q} - \theta_k) \sin \hat{q}] \rho_{\perp} k_{\theta}(v_{\perp}^2 + v_{\parallel}^2/2)/v_{\parallel} \geq 1$ must be satisfied, which is not true, except over a small region of velocity space. Therefore this approach is abandoned here. Instead, the velocity space integral is performed (with a minimum of a priori ordering assumptions) via the "curvature approximation," made by substituting $v_{\perp}^2 \rightarrow 2 v_{\parallel}^2$ in $\omega_{Di}$. This choice is substitution is motivated by the fact that both quantities have the same mean value, and can be expected to have roughly the same general behavior. Quantitative accuracy of the curvature approximation is discussed by Terry et al. With this, the velocity integral in Eq. (5) can be performed, producing

$$\varepsilon_k(\omega) \hat{\phi}_k(\hat{q}) = 0,$$

(6)

where

$$\varepsilon_k(\omega) = 1 + \frac{1}{\tau} + \frac{\Gamma_0}{(a_+ - a_-) \Delta} \left[ \frac{\eta_\parallel}{\eta_\parallel} \right] \left[ \frac{\eta_i}{\eta_i} \right] \left[ \frac{\omega}{\Omega} \right] \left[ \frac{\xi}{\Omega} \right] \left[ \frac{A - \Omega}{A} \right]$$

$$- a_+ Z'(a_-) + \frac{\eta_\parallel}{\eta_\parallel} (1 + \Omega)$$

$$\times [Z(a_+) - Z(a_-)],$$

(7)

and $\eta_{i-1}^{-1} = \frac{1}{b} (1 - \Gamma_{\perp}/\Gamma_0)$ is the $k_\parallel$-dependent threshold in the local slab theory, $Z$ is the plasma dispersion function, $\Gamma_{\perp} < \Gamma_\parallel \sim \eta_i$ (where $\Omega = \omega_{\parallel} + i \Omega_m$), $qR k_{\parallel} \sim 1/\eta_i$, and $b_{\parallel} \gg 1$ (verifiable a posteriori). This solution also applies to flat density profiles, and for any of the following equations it is legitimate to take $L_n \rightarrow \infty$, as long as $\epsilon_i - L_i/R \ll 1$ still holds. Note that with this ordering, the dominant resonances are $\omega = k_{\parallel} v_{\parallel}$. It is possible that there is an alternative ordering in which the drift resonance dominates, but we have not been able to find such a solution to Eqs. (6) and (7). Expanding $a_{\pm}$ and $Z(a_{\pm})$ with this ordering, $\varepsilon_k$ becomes

$$\varepsilon_k(\omega) = 1 + \frac{1}{\tau} + \frac{\Gamma_0}{(a_+ - a_-) \Delta} \left[ \frac{\eta_\parallel}{\eta_\parallel} \right] \left[ \frac{\eta_i}{\eta_i} \right] \left[ \frac{\omega}{\Omega} \right] \left[ \frac{\xi}{\Omega} \right] \left[ \frac{A - \Omega}{A} \right]$$

$$+ i \sqrt{\tau} \frac{k_{\parallel}}{k_\parallel} \left( \frac{\eta_\parallel}{\eta_\parallel} \right) \left( 1 + \Omega - \eta_i \xi^2 \right).$$

(8)

This can be converted into a differential equation by rewriting $k_{\parallel}^2$ as $-(\eta_i) \left[ - \xi^2 \partial^2 / \partial \theta^2 \right]$, and with this Eqs. (6) and (7) become

$$\frac{\partial^2}{\partial \theta^2} \hat{\phi}_k + Q(\Omega) \hat{\phi}_k = 0,$$

(9)

where the "potential function" is given by

$$Q(\Omega) = - b_{\parallel} \xi^2 \frac{\eta_\parallel}{\eta_i} \left[ \frac{\eta_i}{\eta_\parallel} \right] \left[ \frac{\omega}{\Omega} \right] \left[ \frac{\xi}{\Omega} \right] \left[ \frac{A - \Omega}{A} \right]$$

$$+ i \sqrt{\tau} \frac{\xi}{\eta_\parallel} \left( \frac{\eta_\parallel}{\eta_i} \right) \left( 1 + \Omega - \eta_i \xi^2 \right).$$

Equation (9) describes the frequency, growth rate, and parallel mode structure of the linear modes under the above ordering scheme. Note that $Q$ depends on $\partial / \partial \theta$ through the presence of $\xi(k_{\parallel})$, so that formally Eq. (9) is a third-order differential equation. However, such dependence only appears in terms that will be taken as higher order, and thus will be iteratively substituted from the lowest-order solution. The rest of this section is devoted to the solution of Eq. (9).
B. Linear modes

In this subsection we describe the solution of Eq. (9). Since the methods used are not standard, the first part is devoted to describing the analytical techniques used; the remainder examines the modes at high and low parallel mode number.

1. Method of solution

Equation (9) can be solved by taking \( \lambda - \Omega_b / \Omega_k \ll 1 \) (where \( \Omega = \Omega_R + i \Omega_I \)) as an ordering parameter (valid when \( \eta_i \) is sufficiently close to the threshold) and expanding \( \hat{\phi} = \hat{\phi}^{(0)} + \lambda \hat{\phi}^{(1)} \) and \( Q(\Omega) = Q_R(\Omega_R) + i \lambda \{ Q_I(\Omega_R) \Omega_R + \Omega_I \partial Q_R(\Omega_R) / \partial \Omega_R \} \). This allows a perturbation solution for the mode structure, frequency (lowest order), and growth rate (next order).

The frequency \( \Omega_R \) and the mode structure are described by Eq. (9) to lowest order in \( \lambda \), which is

\[
\frac{\partial^2 \hat{\phi}_k^{(0)}}{\partial \Omega^2} + Q_R(\Omega_R) \hat{\phi}_k^{(0)} = 0, \tag{10}
\]

where

\[
Q_R(\Omega_R) = -\frac{b \phi^2 \eta_i \Gamma_0 \left[ k^2(\hat{\theta}) \right]}{\epsilon_n (1 + 1/\tau)} \cos \hat{\theta} + \frac{\Omega_R}{2 \epsilon_n},
\]

with the boundary condition that \( \hat{\phi}_k^{(0)} \rightarrow 0 \) as \( \hat{\theta} \rightarrow \pm \infty \). The WKB solution of Eq. (10) is complicated by the fact that \( Q_R \) has an infinite number of turning points at intervals of about \( \pi \). This means that the usual decaying WKB branch, \( \hat{\phi} = \exp(-|\hat{\theta}| \sqrt{Q} \ d \hat{\theta} / |\hat{\theta}|) \), will pick up a component of the growing branch at each turning point, and thus not satisfy the boundary conditions. Instead, asymptotically vanishing solutions to Eq. (10) can be found in the regions in which \( |\hat{\theta} - \theta_k| > \max[1, \Omega R/2 \epsilon_n] / \beta \), in which case the term with sin \( \hat{\theta} \) dominates the potential and \( \Gamma_0 \left[ k^2(\hat{\theta}) \right] \rightarrow \Gamma_0 \left( k \hat{\theta} / 2 \right) / |\hat{\theta} - \theta_k| \), and \( Q_R(\hat{\theta}) \) becomes

\[
Q_R = -\mu \text{sgn}(\hat{\theta} - \theta_k) \sin \hat{\theta}, \tag{11}
\]

where \( \mu = b \phi^2 \eta_i / \epsilon_n \) and \( \rho = (1 + 1/\tau) / \Gamma_0 \left( k \hat{\theta} / 2 \right) \). Since Eq. (11) represents a periodic potential, then the vanishing eigenmodes are the Bloch functions, \( \hat{\phi} \) for which \( \hat{\phi} \) has the form \( \exp(a \hat{\theta}) f(\hat{\theta}) \), where \( f \) is a function with period \( 2\pi \) (further described in the Appendix). Between these two asymptotic regions is a “central region,” defined as the region where \( Q_R \) is significantly different from Eq. (11). The central region is roughly where \( |\hat{\theta} - \theta_k| < \max[1, \Omega R/2 \epsilon_n] / \beta \), which will be taken as the average domain of the central region (although the boundary is difficult to define exactly). A normal mode extends between the two turning points bordering the central region, and will connect the two asymptotically decaying branches if the following phase quantization condition is satisfied (see the Appendix for a derivation):

\[
\int_{\hat{\theta}_-}^{\hat{\theta}_+} \sqrt{Q_R(\Omega_R)} d\hat{\theta} = \pi \ell, \tag{12}
\]

where \( \hat{\theta}_\pm \) are the turning points bordering the “classically forbidden regions” on either side of the central region [i.e., \( Q_R(\theta_\pm = 0 \) and \( Q_R(\theta_\pm = \epsilon) < 0 \), and \( \ell \) is a non-negative integer. As for the structure of the mode within the central zone, when \( Q_R > 0 \) the WKB solution for \( \hat{\phi}_k^{(0)} \) is, up to a phase,

\[
\hat{\phi}_k^{(0)} \propto \sin \left[ \frac{\int \sqrt{Q_R} d\hat{\theta}}{Q_k^{1/2}} \right]. \tag{13}
\]

Figure 1 shows a shooting code solution of the \( l = 4 \) mode, showing the localization of the mode to the central region and decay outside this. For some modes, there will be parts of the central zone where \( Q_R < 0 \) (making \( \hat{\phi}_k^{(0)} \) exponential in \( \hat{\theta} \)), but these regions will be less prevalent than the oscillatory regions described by Eq. (13). This is because (a) the tendency for \( \Omega_R > 0 \) (propagation in the ion diamagnetic drift direction) has the effect of making \( Q_R \) positive over the majority of the central region, and (b) every exponential solution must be matched on either side to an oscillatory solution of the form in Eq. (13), so that the amplitude in a given exponential zone will less than that of the neighboring oscillatory zone.

The growth rate is obtained from taking Eq. (9) to first order, which is

\[
\frac{\partial^2 \hat{\phi}_k^{(0)}}{\partial \Omega^2} + Q_R(\Omega_R) \hat{\phi}_k^{(0)} = 0, \tag{14}
\]

where

\[
Q_I = -\left( \mu \Omega_R / 2 \epsilon_n \eta_i \xi \right) Z_i(\xi) \eta_i / \xi (\eta_i \eta_c - 1 + \Omega_i - \eta_i \xi^2), \tag{15}
\]

and \( Z_i = \sqrt{\pi} e^{-\xi^2} k_i / k_\epsilon \). Multiplying Eq. (14) by \( \hat{\phi}_k^{(0)} \), and taking \( \int_{-\infty}^{\infty} d\hat{\theta} \) annihilates the \( \hat{\phi}_k^{(0)} \) terms, and yields the following expression for the growth rate:

\[
\hat{\phi}_k^{(0)} \propto \sin \left[ \frac{\int \sqrt{Q_R} d\hat{\theta}}{Q_k^{1/2}} \right]. \tag{13}
\]

**FIG. 1.** Typical solution to Eq. (10), showing the potential, \( -Q_R \) (dashed line), and the solution, \( \hat{\phi} \) (solid line). Here \( l = 4, \Omega_R = 0.103, b = 1, \epsilon_\perp = 0.5, \theta_k = \pi, q = 1, \eta_i = 1, \hat{\theta}_1 = 1, \text{and } \hat{\theta}_2 = 1. \)
The threshold at a given $k$, denoted as $\eta_c^\text{th}(k)$, can be estimated directly from Eq. (16) (without the need to calculate $\Omega_f$ first) by setting the numerator to zero, which produces

$$\eta_c^\text{th}(k) = \eta_c \frac{\int_{-\infty}^{\infty} d\tilde{\theta} \left[ 1 + \frac{\Omega_R}{2\tilde{\theta}_0} \int_{\tilde{\theta}_0 + \Delta\tilde{\theta}}^{\tilde{\theta}_0 - \Delta\tilde{\theta}} d\tilde{\theta} \left( \frac{\Omega_R}{2\epsilon_s} \frac{Q_R}{\mu} - 1 \right) \right]}{\int_{-\infty}^{\infty} d\tilde{\theta} \left( \frac{\Phi_0(\tilde{\theta})}{\xi} \right)^2 / \xi} \, . $$

where $\eta_c(k) = \eta_c(k_0)$ has been used, valid for $k_0 \gg 1$. Equation (17) can be simplified by (a) approximating $\phi$ by Eq. (13), with a width of $\Delta\tilde{\theta}$, (b) using the WKB approximation for $k_n$, so that $k_n^2 = Q_R/\mu^2$, and (c) noting that $\Phi_0(\tilde{\theta})$ oscillates much faster than $Q(\tilde{\theta})$, so that the $\sin^2(k_n\tilde{\theta})$ term coming from $(\phi_0)^2$ can be replaced with $1$. These approximations produce

$$\eta_c^\text{th}(k) \approx \eta_c \left[ 1 + \frac{\Omega_R}{2\tilde{\theta}_0} \int_{\tilde{\theta}_0 + \Delta\tilde{\theta}}^{\tilde{\theta}_0 - \Delta\tilde{\theta}} d\tilde{\theta} \left( \frac{\Omega_R}{2\epsilon_s} \frac{Q_R}{\mu} - 1 \right) \right] \, . \tag{18}$$

Thus we have methods for evaluating $\Phi_0(\tilde{\theta})$, $\Omega_R$, and $\eta_c$. The radial envelope of the linear modes is described by a higher-order equation in $d^2\omega_r / dr^2$, but such detail is not necessary here, where radial structure is introduced nonlinearly (see the second paragraph of Sec. V).

2. High $l$ limit

When $l$ is high enough, $\Omega/2\epsilon_r$ is the dominant term in $Q_R$, over most of the central region. In this limit, the phase quantization integral in Eq. (12) may be estimated by taking $Q_R = \mu \Omega_R / 2\epsilon_r$, and $\Omega > \epsilon_r = \epsilon_r + \Omega_R / 2\epsilon_s, [\text{obtained from Eq. (10) by balancing the $\sin \tilde{\theta}$ and $\Omega_R$ terms}].$ This yields

$$\Omega_R = 2\epsilon_r (\tilde{\theta}_0^2 / \mu^2)^{1/4} \, . \tag{19}$$

Equation (19) can be compared with the shooting code trace of $\Omega_R$ vs $k_0\omega_r$ shown in Fig. 2. In order to satisfy the assumption $\Omega_R / 2\epsilon_r \gg 1$, Eq. (19) leads to the condition that $l \sqrt{\mu} \tilde{\theta} > O(1/\tilde{\theta}_0)$. The high $l$ modes extend across the central region, decaying exponentially outside this, and thus are centered around $\tilde{\theta} = \tilde{\theta}_0$, with a width of approximately $(\tilde{\theta}_0^2 / \mu^2)^{1/4}$. Within this width, $\Phi_0(\tilde{\theta}) = \sin(\tilde{\theta}_0 \tilde{\theta})$, where $k^2 = \mu \Omega_R / 2\epsilon_r$.

The growth rate can be evaluated using Eq. (16), taking $Q_R = \mu \Omega_R / 2\epsilon_r$ across the width of $\Phi_0(\tilde{\theta})$. This yields a linear growth rate of

$$\Omega_f = \left[ \frac{\pi \epsilon_r}{\mu} \left( \frac{\tilde{\theta}_0^2}{\mu} \right)^{1/6} \times \left[ \frac{\eta_i}{\eta_c} - 1 - 2\epsilon_r (p - 1) \left( \frac{\tilde{\theta}_0^2}{\mu} \right)^{1/3} \right] \right] \, . \tag{20}$$

Equation (20) shows that $\eta_i^{\text{th}}$ decreases with $l$, implying that the low $l$ limit is the most relevant in the threshold regime.

3. Low $l$ limit

Since, from above, only the lowest $l$ are unstable near the threshold, it is important to note that the lower limit of $l$ is $l \gtrsim \mu \tilde{\theta}_0$, not zero, as is often the case. To see this, note that as $l$ decreases from the high $l$ limit considered above, so also do the frequency $\Omega_R$, the wave number, $k_0 = Q_R / \sqrt{\epsilon_r}$, and the mode width, $\Delta\tilde{\theta} = \max \left[ 1, \Omega_R / 2\epsilon_r, \right]$. This continues until $\tilde{\theta}_0$ and $k_0$ reach their lower limits of $\tilde{\theta}_0 = 1/\sqrt{\epsilon_r}$ and $k_0 = 1/\sqrt{\epsilon_r}$. At this point $l$, which must obey the relation $l = k_0 \Delta\tilde{\theta}$, cannot decrease further, and so $l \gtrsim \mu \tilde{\theta}_0$.

Unfortunately, the low $l$ modes are difficult to treat analytically, since they have $\Omega_R / 2\epsilon_r \sim \cos \tilde{\theta} = 1$, so that $Q_R$ cannot be easily ordered. However, the following general observations can be made.

1. The real part of the frequency is determined by the phase quantization condition

$$\int_{\tilde{\theta}_0 - 1/\sqrt{\mu}}^{\tilde{\theta}_0 + 1/\sqrt{\mu}} d\tilde{\theta} \sqrt{\frac{\Omega_R}{2\epsilon_r} - \cos \tilde{\theta}} = \frac{\pi l}{\sqrt{\mu}} \, .$$

The solution of this involves inversion of an elliptic integral of the second kind, but the general features of the solution can be obtained by considering the case, where $l$ is just above its lower cutoff. In this case the solution can be found by expanding the radical for $\Omega_R / 2\epsilon_r > \cos \tilde{\theta}$, performing the integral, and solving by iteration, which produces

$$\frac{\Omega_R}{2\epsilon_r} = \left( \frac{3\pi l}{2\sqrt{\mu}} \right)^2 + \sqrt{\mu} \left( \frac{3\pi l}{2\sqrt{\mu}} \right) \sin \left( \frac{1}{\sqrt{\mu}} \right) \cos \tilde{\theta}_0 \, .$$

For higher $l$, the first term dominates, but as $l$ decreases to its lower limit of $\sqrt{\mu} / 3$ then both terms become of order 1, after
which the mode is cut off. The cutoff point is difficult to
determine analytically, and the value of $\Omega_r$ at this point is
complicated by the quantization of $l$, but it is sufficient for
the present purposes to characterize the frequency of these
lowest $l$ modes as

$$\Omega_r \approx \frac{e \epsilon}{2} \pm O(\epsilon_c).$$  \hspace{1cm} (21)

This value and the high level of dispersiveness are verified by
shooting code solutions to Eq. (10). (Figure 2 shows a typi-
cidual numerical dispersion curve. Note that the termination of
lower $l$ branches with increasing $b$ is a real physical effect.
The apparent divergence of $\Omega_r$, as $b \rightarrow 0$ is a by-product of
the standard $\omega/\omega_p$, $\rho$, normalization, and also the ordering
of $k\rho$, $l$, $\delta$, and other parameters. We have not attempted to follow the details of the
individual modes, but we expect that the theory is a fairly
good representation of the average collective behavior (scal-
ing, ordering, etc.) of the linear modes.

The modes considered here have several differences from those usually discussed\[19,22\] in the context of the toroidal $\eta_t$ threshold. It would be premature to say here whether one
solution is more correct than the other, since a different part of $k\rho$ space has been considered \[k\rho \rightarrow 0\] in Ref. 19,
while $k\rho \sim O(1)$ here, and also the present analysis applies
only insofar as the WKB approach is valid. However,
the previous picture would benefit from a derivation that
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The preceding linear theory only roughly represents the
complicated behavior of the exact solution. In reality, there are
a large variety of possible modes, and the details of any
given mode depend sensitively on $k\rho$, $\epsilon$, $l$, $\delta$, and other pa-
rameters. We have not attempted to follow the details of the
individual modes, but we expect that the theory is a fairly
good representation of the average collective behavior (scal-
ing, ordering, etc.) of the linear modes.

performed by expanding the denominator of the integrand in Eq. (5) for \( \omega_D \ll \omega - k_i v_b \). (For \( |\hat{\theta} - \theta_0| > 1/\sqrt{\epsilon_r} \), \( \omega_D > k_i v_{ti} \), making this expansion invalid, but this occurs well outside the limits of the mode, which has \( |\hat{\theta} - \theta_0| \leq 1/3 \sim 1 \). This leads to an equation similar to Eq. (9), but the curvature approximation is preferred because it avoids the need for such an a priori ordering assumption at this point. Second, the weak instability ordering that allows Eq. (9) to be divided into Eqs. (10) and (14) requires that \( \gamma < \omega_r \), taking \( \Omega_R \approx 2\epsilon_r \), and Eq. (26) gives the requirement

\[
\eta_i - \eta_{\text{threshold}}^\text{max} < 2\epsilon_r \sqrt{1 + (1/\tau)} / \epsilon_r. \tag{27}
\]

Other restrictions of the theory include \( k_y \geq O(1) \), \( \epsilon_T \ll 1 \) (although \( \epsilon_T \sim \omega \) is permissible, with slight redefinitions of various parameters). Third, validity of the WKB approximation and the local treatment of \( k_y \) require that the scale length of \( Q_{\text{av}} (\hat{\theta}) \) (order of 1) is longer than that of \( \phi (\hat{\theta}) \) (which from Eq. (22) is of order \( \sqrt{\epsilon_r} \)). Thus the WKB approximation applies in the limit \( \epsilon_T \ll 1 \). The analytical solution for \( \phi (\hat{\theta}) \) and \( \Omega_R \) has been tested with a shooting code solution of Eq. (10). Figure 1 shows a typical mode, and Fig. 2 shows \( \Omega_R \) as a function of \( k_y \). It is found that the analytical solution describes well the average behavior in a gross sense (i.e., scalings, orders of magnitude, cutoff at low \( l \), average mode width, \( k_y \), etc.), but the complicated behavior of the exact solution means that the details of a mode at a given \( k \) might deviate somewhat. Finally, it would be desirable to compare the analytical prediction of the growth rate with numerics, but this cannot be done with the simple shooting method used here. The difficulty lies in the dependence of \( \gamma \) on \( \text{Im } Z \) in Eqs. (15) and (16), which in turn varies with \( k_y / k_y \). This term is easily handled in the WKB approximation (where \( k_y \) is treated as a number), but presents difficulty in the context of a shooting code (where the operator \( V_{\text{av}} \) is used instead). It should be possible to check the existence and growth of these modes with existing gyrokinetic codes. However, such studies have concentrated either on the \( l = 0 \) branch or the \( k_y \rho_i \ll 1 \) limit of the mode. The modes examined here, with \( l \sim R / L_k \) and \( k_y \rho_i \gtrsim 1 \), have apparently not been addressed, although the present study suggests that they can be quite unstable near threshold. Furthermore, these modes are interesting since they provide a paradigm to study the effects of nonlinear Landau damping on the toroidal \( \eta_i \) mode, which may be important for all branches of the mode.

IV. NONLINEAR THEORY

In this section we examine the nonlinear saturation of the linearly unstable modes just above the threshold, the linear properties of which are described in Sec. III B 3 above. A weak turbulence expansion is used to derive the wave kinetic equation, which is then solved to yield the turbulent spectrum. This expansion requires that the nonlinear terms be less than the dominant linear term, which is equivalent to the threshold orderings already employed in the linear theory (\( \gamma < \omega_r \)). To facilitate this analysis, we use a local analysis in this section, whereby \( k_y \) is treated as a number in the wave kinetic equation (although the results from the normal mode analysis regarding the structure of \( \phi (\hat{\theta}) \) are retained).

Application of the weak turbulence expansion to Eq. (3) proceeds by the same method as described in Ref. 14, and is not repeated here. To lowest order, this expansion yields Eq. (10) describing the linear mode structure and frequency, and to next order it yields the wave kinetic equation, which describes the evolution of the spectrum due to linear instability and also due to two nonlinear processes. The first nonlinear process is three wave resonance, which is negligible since wave triplets that obey both \( k = k^0 + k^\nu \) and \( \omega_k = \omega_{k^0} + \omega_{k^\nu} \) are rare (due to the low level of frequency broadening in the weak turbulence regime combined with the strong dispersion of \( \omega_k \)). The second saturation process is ion Compton scattering (which occurs when an ion resonates with the beat wave between two normal modes), which can be expected to accompany the presence of strong linear ion resonances. This is the dominant saturation mechanism for the threshold regime. To simplify notation, it is convenient to represent the nonlinear gyrokinetic equation, Eq. (3), as

\[
- iL_{1,k} (\omega_k) \hat{h}_k - iL_{2,k} (\omega_k) \hat{p}_k = \sum_{\rho' \sigma'} \int dk' c_{k' k} \phi_{k'} \hat{h}_{r' r},
\]

where the \( L \) represents the linear coefficients and the \( C \) represents the nonlinear coupling coefficient. With this, the wave kinetic equation (less the three wave resonance terms) can be written locally as

\[
\frac{1}{2} \frac{\partial}{\partial \omega_r} \left( \frac{\partial}{\partial t} \phi_k (\hat{\theta}) \right) = - \text{Im } \epsilon_k (\omega_r) \phi_k^2 (\hat{\theta}) - \text{Im } \sum_n \int \! dk' \epsilon_k^{(3)} (\omega_r) \phi_k^{(2)} (\hat{\theta}) \left| \phi_k (\hat{\theta}) \right|, \tag{28}
\]

where \( \epsilon_k \) is the linear dielectric function given by Eq. (7),

\[
\epsilon_k^{(3)} (\omega_{k'} - k, k, k) = \frac{1}{2} \int \! d^3v J_0 (k_{1,1}) L_{1,k} (\omega_k) C_{k', k'} C_{k', k', k}, \tag{29}
\]

\( \omega_k = \omega_{k'} - \omega_k \) (not \( \omega \) of the beat wave), \( k'' = k - k' \) and \( \epsilon_k \) is given by Eq. (8). The coupling of \( p \) is negligible for \( k > 1/\pi \) (since the quasimodes have width \( \Delta_k \lesssim 2/\pi \)), and has been left out of Eq. (28) (although the results would probably not change much otherwise). Also, there is a term in \( \epsilon_k^{(3)} \) that has canceled under \( \Sigma_k \) by symmetry. Taking \( \int \! d^3 \phi \phi \) of Eq. (28) reproduces Eq. (16) for the linear growth rate (with an additional nonlinear term). Here, we neglect this averaging process for simplicity, using instead quantities which reflect average behavior across the width of \( \phi_k \). Consistent with this approach, \( \epsilon_k \) will be kept in the local form given by Eq. (8).

We next focus on reducing \( \epsilon_k^{(3)} \) to a more tractable form. The calculation is facilitated by the fact that the \( \omega_{k' \nu} \) terms give only higher-order contributions to resonances, given the ordering of the linear theory. Thus, taking \( L_{1,k} (\omega_k) - \omega_k - k_{1,1} v_{ti} \) for the lowest-order contribution to
the resonance allows the denominator in Eq. (29) to be written as the sums of single resonances

\[
\frac{1}{L_{1,k} (\omega_k)^2 L_{1,-k} (-\omega_k^*)}
\]

Applying this to Eq. (29) yields

\[
\text{Im} \epsilon^{(3)}_{k',-k',k} = -\frac{\alpha}{\omega_k (2\pi \delta k \phi k')^2} \left[ \left( \frac{\partial}{\partial \phi} - \theta_k \right)^2 \left( \frac{\partial}{\partial \phi} - \theta_k^* \right)^2 \left( \frac{\partial^2}{\partial \phi^2} \right) \right]
\]

and any terms of relative order \( \sqrt{\zeta} \) (by the linear ordering) have been dropped. In evaluating the contribution from parallel resonances, analytic continuation has been applied (yielding terms like \( k_k\ell /k_k\ell \)), which changes the symmetry properties of \( \epsilon^{(3)} \), and has an impact on the nature of the turbulent energy transfer (discussed near the end of this section).

The wave kinetic equation, Eq. (28), can now be written as

\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\psi e^{-a^2/2} \left( \frac{k_k^2}{L_{1,k} (\omega_k)} \right) - \frac{k_k^2}{L_{1,k} (\omega_k)} \frac{\partial}{\partial \omega_k} \left( \frac{\partial}{\partial \omega_k} \right) \mid _{\delta = 1} = -\sqrt{\pi} \omega_k (2\pi \delta k \phi k')^2 \left[ \left( \frac{\partial}{\partial \phi} - \theta_k \right)^2 \left( \frac{\partial}{\partial \phi} - \theta_k^* \right)^2 \left( \frac{\partial^2}{\partial \phi^2} \right) \right]
\]

where

\[
G(b,b') = \int_0^1 d(u') f_1' (\sqrt{2b} u) f_1' (\sqrt{2b'} u) \exp(-u'),
\]

\[
G'(b,b') = \left( \frac{3}{2} + \frac{\partial}{\partial \beta} \right) G(b,b') \left| _{\beta = 1} \right.
\]

and any terms of relative order \( \sqrt{\zeta} \) have been dropped. In evaluating the contribution from parallel resonances, analytic continuation has been applied (yielding terms like \( k_k\ell /k_k\ell \)), which changes the symmetry properties of \( \epsilon^{(3)} \), and has an impact on the nature of the turbulent energy transfer (discussed near the end of this section).

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Here, the linear growth rate \( \gamma_k \) can be obtained by using the local form of \( \epsilon_k \), and is given by

\[
\gamma_k = \left( \sqrt{2\pi} |k_k| / \omega_k \right) (\eta_k - \eta_k'^{(\text{or})}),
\]

where \( \eta_k (1 - \Omega_k + \eta_k (\zeta^2) / \pi \zeta \right) \equiv \eta_k'^{(\text{or})} \) has been used. (Note that \( \eta_k \) does not increase indefinitely with \( k_k \), since \( \eta_k'^{(\text{or})} \) has implicit \( k_k \) dependence.) The nonlinear growth can be written by inserting \( \epsilon^{(3)} \) into Eq. (28), and is given by

\[
\gamma_k^N = \frac{8 \pi^{2/3} (3k \phi)^3}{\Gamma_0 (\eta_k)} \int_{-\infty}^{\infty} dk' \sum_{k_n^2} k_n^2 (\theta_{k_k} - \theta_{k_k}^*)^2
\]

\[
\times \left[ \eta_k G'(b_{b_k} b_{b_k}) - G(b_{b_k} b_{b_k}) \right]
\]

\[
\times \left| k_n \right| \Theta (-k_n k_k) \left| \hat{\phi}_{k_k}^2 \left( \Theta \right) \right|,
\]

where \( \Theta \) is the Heaviside step function and \( \zeta^2 = \omega_k / \sqrt{2k_k} \).

In the saturated state, \( \partial \hat{\phi}_{k_k}^2 / \partial t \rightarrow 0 \), and Eq. (31) becomes \( \gamma_k^N = 0 \), giving an equation for the saturated turbulent spectrum. In order to solve this, it is useful to approximate the spectral sums (in \( n' \) and \( k_k \)) as integrals. Making use of the Heaviside step function, the symmetry of

\[
\text{lim \ left[ \text{integrant in Eq. (34)]} \right\}
\]

\[
\kappa_{\zeta} = \text{lim} \ \int_{-k_0}^{k_0} \left( \kappa_{\zeta} (\partial \zeta / \partial \kappa_0) + \Theta_{\kappa_0} (\partial \zeta / \partial \kappa_0) \right)^2,
\]


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which demonstrates that there is a logarithmic divergence resolved only by the weak dependence of $\xi$ on $\theta$. On the other hand, for $k_\theta \to \infty, k_\parallel \to -k_\parallel$, and we have

$$\lim_{k_\parallel \to \infty} \text{integrand in Eq. (34)} \propto |\delta_{k_{\parallel}}|^{-2} \sqrt{|k_\parallel|},$$

which converges well if the spectrum vanishes faster than $1/|k_\parallel|$. Thus the $k_\parallel$ integral can be performed by keeping only the dominant contribution from the lower limit, for which $k_\parallel = k_\parallel$, and Eq. (34) becomes

$$\eta_\parallel - \eta_\parallel^{\text{inv.}} = \frac{2 \sqrt{2} (2\pi)^{3/2} \frac{\eta_\parallel}{k_\parallel}}{R_\parallel} \frac{k_\parallel}{k_\parallel} \times \left[ \eta_\parallel G'(b_{\parallel}, b_{\parallel}) - G(b_{\parallel}, b_{\parallel}) \right] |\hat{\phi}_k|^2$$

$$\times \int dk_{\parallel}^{1/2} a \frac{|k_{\parallel}| \theta_{\parallel}^{2} k_{\parallel}}{\xi_{\parallel}^{2}}.$$  \hspace{1cm} (35)

The $\theta_{\parallel}$ integral is performed by keeping only the $\theta_{\parallel}^{2}$ dependence in the integrand (thereby taking advantage of the relative flatness of the spectrum in $\theta_{\parallel}$). This leads to the following spectrum:

$$|\hat{\phi}_k|^2 = \frac{1}{(2\pi)^2} \frac{\eta_{\parallel} - \eta_{\parallel}^{\text{inv.}}(k)}{R^2} \frac{1}{k_{\parallel}^{2}}, \hspace{1cm} (36)$$

and $|\hat{\phi}_k| = 0$ in regions of $k$ space, where $\eta_{\parallel} < \eta_{\parallel}^{\text{inv.}}(k)$. The logarithmic term in the denominator represents a correction of order unity, and since $\theta_{\parallel} / \theta_{\parallel}$ is not calculated in this theory, we shall neglect it.

It is useful to convert the quasimode spectrum of Eq. (36) into a Fourier spectrum, since this is what would be measured experimentally. The primary difference between the two is the radial wave number, which is constant for a Fourier mode, but varies along the parallel direction of a quasimode. The relation between the two spectra can be obtained from the Fourier transform of Eq. (1), which produces

$$\hat{\phi}_k = \int_{-\infty}^{\infty} d\theta \ e^{i(m - n)\theta} \frac{\varphi_{\theta}^{m,n}(k)}{k_{\parallel}} \Theta(|k_{\parallel} - k_{\parallel}^{0}|)$$

$$= \hat{\phi}_{\parallel} \Theta(|k_{\parallel} - k_{\parallel}^{0}|) \delta_{m - n} g_{k_{\parallel}}, \hspace{1cm} (37)$$

where $k$ is the radial wave number of the Fourier modes, $k_{\parallel}$ is the parallel wave number in terms of the ballooning coordinate, $\Theta$ is the Heaviside step function, and in the second step of Eq. (22) has been used for $\hat{\phi}^{(\theta)}$. The Kronecker delta function describes the localization of the real space fluctuations to the rational surfaces. In a more careful treatment, the parallel wave number $k_{\parallel}$ would be a wave packet, and the delta function would become an envelope describing the radial width of a fluctuation. Equation (37) shows that the Fourier spectrum is similar to the quasimode spectrum in $k_{\parallel}$, and with a (roughly) flat spectrum of radial wavelengths of width $k_{\parallel}$. Note that a more complete theory would predict a radial envelope for $\hat{\phi}_{\parallel}$, giving the radial Fourier spectrum a cutoff as $k_{\parallel} - 0$.

Compared with the mixing length level of $\delta$ generally invoked for strong fluid toroidal $\eta$ turbulence, Eqs. (36) and (37) are smaller by roughly a factor of $L_{\parallel}/R \lesssim 1$, implying that nonlinear ion resonances are a strong damping mechanism for toroidal $\eta$ turbulence. However, the difference in saturation levels here is not as pronounced as in the slab model, where a factor of $(L_{\parallel}/R)^2$ was found.14 The difference from the slab model seems not to arise from any fundamental difference in the physics, but rather from the different linear ordering of $\omega$ and $k_{\parallel}$ in the two models. These points are discussed further at the end of Sec. V.

It appears that this nonlinear saturation process (involving two waves and a particle) transfers energy directly from the spectrum to the particle distribution, without conserving turbulence energy. This can be seen by noting the following points from Eq. (30) (which is not approximated beyond application of the weak turbulence expansion, the eikonal approach, and the ordering from the linear theory).

1. The nonlinear growth of $\phi_{\parallel}$ comes only from interaction with the $\delta_{\parallel}$, that have $k_{\parallel}$ larger than $k_{\parallel}$ (owing to the presence of the Heaviside step function, with $k_{\parallel} = k_{\parallel} - k_{\parallel}$).

2. The nonlinear growth rate is negative definite for $\eta, G_{\parallel,k}, G_{\parallel,k} > 0$ (true whenever the linear instability criterion is satisfied).

Point (1) implies that in the nonlinear interaction between $\delta_{\parallel}$ and $\delta_{\parallel}$ only the former changes energy, and the only place for the energy to go is the distribution function. Furthermore, $\phi_{\parallel}$ must be damped by this process, by point (2) above. Thus the nonlinear process transfers energy from the mode with lower $k_{\parallel}$ to the resonant ions, and the second mode plays the role of a nonlinear catalyst that neither gains nor loses energy. This means that an $\eta_\parallel$ instability saturated by nonlinear Landau damping does not require a part of $k$ space where the modes are linearly stable; the nonlinear process itself provides an energy sink. It appears that this picture of nonlinear Landau damping also applies to the slab limit of the $\eta_\parallel$ instability, although a previous work did not uncover this fact, because of the less detailed treatment of the nonlinear scattering term. Specifically, the previous work did not apply analytic continuation to the resonant denominator, Eq. (29), which leads eventually to the Heaviside step function in Eq. (33) and has a significant impact on the symmetry properties with respect to $k$ and $k'$.

It is well known that drift waves can also saturate by nonlinear wave–particle interaction, except that it is generally found that this process transfers energy to waves, not to the particle distribution. The underlying difference can be seen from the relative ordering of the phase and thermal velocities, which leads to an adiabatic constraint for drift waves, but not for $\eta_\parallel$ modes. For drift waves, the linear time scale is characterized by $\omega \gg k_\parallel v_\parallel$, while nonlinear wave–particle interaction requires $\omega - \omega = (k_\parallel - k_\parallel) v_\parallel \sim k_\parallel v_\parallel$. The nonlinear interaction is thus slow relative to the linear time scales $(\omega \gg |\omega' - \omega|)$, and can be regarded as an adiabatic perturbation on the waves, which conserves wave action density through the well known Manley–Rowe relations. However, in the case of $\eta_\parallel$ modes near the threshold $\omega \lesssim k_\parallel v_\parallel$, and so there is no such adiabatic constraint on energy transfer, thus allowing the particles to acquire a large part of the energy, as described in the preceding paragraph.
The distribution function).

described here (suggesting direct transfer of wave energy to coming to outgoing waves), and "nonlinear Landau damp-
scattering" seems more appropriate for the drift wave satu-
ration process (suggesting elastic energy transfer from in-
coming to outgoing waves), and "nonlinear Landau damp-
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ration process (suggesting elastic energy transfer from in-
coming to outgoing waves), and "nonlinear Landau damp-
scattering" seems more appropriate for the drift wave satu-
ration process (suggesting direct transfer of wave energy to
the distribution function).

V. TRANSPORT

Having obtained the saturated turbulent spectrum, the associated ion thermal transport due to \(E \times B\) convection may now be calculated. This simplified model does not allow calculation of particle or electron heat fluxes (since electrons have been taken as adiabatic), or of momentum flux (since there is no equilibrium shear flow), but these phenomena may be considered as secondary to the basic dynamics of \(\eta_i\) turbulence, which concerns relaxation of the ion temperature gradient.

Before proceeding with the calculation, it is important to discuss the diffusive nature of the transport. This is an issue because the radial width of the quasimodes is very large\(^{24,25}\) (in fact, infinite to the order considered here), and thus each individual linear mode represents a broad convective cell, implying nondiffusive transport. The variation of the local \(k_s(\theta)\) of a quasimode does not change the situation, since this merely represents the fact that the convective cell twists as it follows the sheared field lines. What does change the situation is the superposition of another mode with a different \(k_s\), i.e., one centered farther down the field line with a twist relative to the first mode. The beat mode between these two (and ultimately many modes) has an oscillatory radial structure, thus breaking the smooth convective pattern of any one mode. Thus the \(k_s\) spectrum (generally not considered in toroidal theories, which tend to assume a single maximally unstable \(k_s\)) is important for giving radial structure to the fluctuations, decorrelating a fluctuation in a radial distance of \(1/3k_0\Delta_\rho \sim \rho_i\) (where \(\Delta_\rho\) is the parallel width of a quasimode). This reconciles diffusive transport with the broad radial width of the linear modes. The situation is analogous to homogeneous Navier–Stokes turbulence, where a given plane wave extends over the entire domain of the fluid; but this long range correlation is broken nonlinearly by the random superposition of many plane waves of different \(k\).

The radial ion thermal flux \(q_i\) from the turbulent \(E \times B\) drift is given by

\[
q_i = \frac{c}{B} \int d^3v \frac{m_i v^2}{2} \langle h(x) \nabla \phi(x) \rangle, \tag{38}
\]

where \(\langle \cdots \rangle\) represents an average over the fast fluctuations. Introducing the ballooning transform via Eq. (2), and carrying out the averages, we find

\[
q_i = (2\pi)^3 \Re \int d^3v \frac{m_i v^2}{2} \int dk_\perp \sum_{\ell \neq 0} \frac{1}{2\pi} e^{2\pi i \ell (p - \rho') - (ik_\| k_{\|}}
\times \left[ \hat{h}_{\perp}(\theta + 2\pi \rho') \hat{\mathcal{h}}_{\perp}(\theta + 2\pi \rho') \right]. \tag{39}
\]

As in the nonlinear theory, the \(p\) and \(p'\) sums can be neglected for \(\xi \approx 1/\pi\) (with assumedly not much difference; even when this condition is not met), and the \(\theta\) average neglected since \(\theta\) dependencies of the integrand are higher order. In the weak turbulence regime, \(h\) is dominated by the quasilinear contribution, and so substituting \(\hat{h}_{\perp}(\theta)\) from the linearized gyrokinetic equation and performing the velocity space integration, after some calculation, produces

\[
q_i = -\frac{(2\pi)^3 m_i}{L_n} \int dk_\perp \sum_{\ell \neq 0} \frac{k_\perp^2 \Gamma_{0\ell}}{|k_{\|}|} \hat{h}_{\perp}^2, \tag{40}
\]

where the approximation \(\Gamma_{0\ell} \approx \Gamma_{0\ell} = 1/\sqrt{2\pi b_0}\) has been applied (valid over the excited part of the spectrum, which has \(b_0 \approx 1\)). Inserting the spectrum from Eq. (36), and \(k_{\|}\) from the linear theory, we obtain

\[
q_i = -m_i \sqrt{1 + \frac{1}{r \left( \eta_i - \eta_{\text{thresh}} \right) \sqrt{L_T / R} / L_n}}, \tag{41}
\]
or an ion thermal conductivity of

\[
\chi_i = \frac{q_i}{\nabla \cdot T_i} = \sqrt{1 + \frac{1}{r \left( \eta_i - \eta_{\text{thresh}} \right) \sqrt{L_T / R} / L_n}. \tag{42}
\]

Compared with the mixing length level often invoked\(^{19,29}\) for strong toroidal \(\eta_i\) turbulence, the above \(\chi_i\) is smaller by a factor of about \(L_T / R\) (as expected from the reduction of spectral amplitude through nonlinear Landau damping). Even with this reduction, \(\chi_i\) is in the range of experimental values.\(^{34}\) This contrasts with a previous finding in the slab theory that Compton scattering reduces \(\chi_i\) by a factor of about \(L_T / R\)^2, making it far too low to contribute to tokamak transport (see the next paragraph for a further discussion of this). In the toroidal theory, it is quite possible that the weak turbulence \(\chi_i\) is large enough to give substantial transport in tokamaks.

It is useful to devise a rule of thumb for estimating \(\chi_i\) on the basis of the linear theory, since the usual mixing length estimate clearly does not approximate Eq. (42) well (nor is it expected to, since it is based on orderings from fluid strong turbulence). This can be done by first estimating the saturated fluctuation level from Eqs. (31) and (33), which produces

\[
\sum_{k_{\perp}} |\hat{h}_{\perp}|^2 = \frac{\xi^2 k_{\|}^2 \gamma_k}{\eta_i k_{\|}^2 \xi^2 \Gamma_0}, \tag{43}
\]

where approximations such as \(\xi \approx \xi' \sim \xi^{\prime\prime}\) have been made, which can be expected to preserve the scalings, if not the exact numerical coefficients. Inserting Eq. (43) into the quasilinear formula, Eq. (40), gives the estimate

\[
\chi_i = \xi^2 \gamma / k_{\|}^2. \tag{44}
\]

This estimate accounts for the different kinetic reduction factors predicted by the slab and toroidal theories, where \(\xi^2 \approx (L_T / L_i)^2\) and \(\xi^2 \sim L_T / R\), respectively. It is not clear how general this formula is, since it is partly based on the linear ordering and weak turbulence expansion specific to the present theory. Equation (44) does, however, make a smooth transition to the usual mixing length estimate, and one might hypothesize that \(\chi_i = \gamma / k_{\|}^2\) applies when \(\xi > 1\).
(fluid behavior), and is replaced by Eq. (44) when $\zeta < 1$ (kinetic behavior). It would be interesting to study the generality of this prescription, but this is beyond the scope of the present work.

VI. CONCLUSIONS

In this work we have examined the linear and nonlinear dynamics of the toroidal $\eta_1$ mode near the threshold of instability, and the resulting transport. Results indicate that inclusion of the parallel dynamics of toroidal $\eta_i$ modes can produce a threshold regime that is more similar to that of the slab $\eta_1$ mode than generally thought. (It is important to note that modes with $k_a\beta_i \gg 1$ are considered here, in contrast to the more usual $k_a\beta_i \ll 1$ assumption.) Linear stability is determined primarily by the balance of sound wave destabilization and damping from transit resonances ($\omega = k_y v_y$), with smaller toroidal corrections, yielding a threshold that varies as $\eta_i^{\text{crit}} = \eta_i^{\text{shb}} + O(\epsilon_1)$. Note that neglecting $k_y$ produces a similar threshold, but predicts entirely different dynamics from those given here, which is especially important for the nonlinear dynamics and transport. Nonlinear saturation is provided by nonlinear Landau damping, and the resulting saturated fluctuations are smaller than the mixing length level by a factor of approximately $L_T / R$. The associated ion thermal transport is reduced by a similar factor, but is still large enough to cause significant transport on tokamaks. Results indicate that $\chi \approx \eta_i^{\text{crit}} g / k_i^2$ is possibly a replacement for the more usual $\chi = g / k_i^2$ in situations where the instability is saturated by nonlinear Landau damping.

As $\eta_1$ rises above this weak turbulence regime (where $\xi \sim \sqrt{\epsilon_T}$), resonances will continue to be important until $\xi \approx 1$. Above some value of $\eta_i$, (not addressed here), fluidlike modes begin to emerge, for which ion resonances can be expected to be less important. However, the higher $l$ eigenmodes (which are often more unstable) are generally more susceptible to resonance (since they have higher $k_y$), and so it is possible that the transport from $\eta_1$ turbulence is sensitive to the effects of nonlinear Landau damping at all values of $\eta_i$. It is also possible that nonlinear resonances persist even when linear resonances become unimportant, as in the case of drift waves. Given that kinetic saturation can quite clearly give important corrections (orders of magnitude) to the transport, and that it is ignored in all theories which invoke either fluid assumptions or mixing length arguments, one might conclude that assessment of this effect is one of the major uncertainties facing reliable predictions from $\eta_i$ mode theories.

It is desirable to compare the ion thermal conductivity given by Eq. (42) with experiments. The restriction $\eta_1 - \eta_i^{\text{crit}} \lesssim O(L_T / \sqrt{RL_T})$ means that experimental agreement is not expected over all parameter regimes, but it is possible that there will be restricted agreement in regions where $\nabla T$ is near the threshold. Possibly an experimental test of the present theory is to see whether Eq. (42) serves as a lower limit for the experimental $\chi_i$ [since the true $\chi_i$ can be expected to exceed Eq. (42) for $\eta_i$ above the weak turbulence regime]. Note that although this weak turbulence theory is formally analyzed for narrow band turbulence (dominated by the linear dispersion relation), in practice such turbulence would be indistinguishable from broadband turbulence, due to the complicated nature of the linear dispersion relation (for example, see Fig. 2).

There are several basic issues addressed in this paper which could be useful in contexts beyond the present regime. First, the linear eigenmode solution (described in Sec. III B 1 and the Appendix) is apparently a new method of solving mode equations that can arise in toroidal geometry, wherein the mode near $\delta = \delta_0$ is matched asymptotically to the Bloch eigenfunctions. The resulting Bohr–Sommerfeld phase quantization condition, Eq. (12), can be applied to any situation where the potential asymptotically approaches a periodic function. Compared with the more usual method of expanding the potential around some local minimum, this method can give significant corrections to the width and number of possible modes. (For example, in this work important corrections to the shear scaling are obtained.) A second observation here is that nonlinear Landau damping can transfer energy directly to the ion distribution, not conserving wave energy. This nonconservative nonlinear energy transfer can provide some fairly counterintuitive phenomena, such as the possibility of a saturated spectrum that does not need to couple to stable modes. Other implications arise regarding the relative role of turbulence in heating versus transport, but this is beyond the scope of the present paper. Finally, the present prescription for reconciling of the broad radial extent of toroidally coupled eigenmodes with diffusive transport (described at the beginning of Sec. V) could have broad applications. This scenario can be extended from weak to strong turbulence if one allows that an arbitrary turbulent field in sheared toroidal geometry can be represented as a sum of quasimodes with approximately random phase. (That the ballooning transform is a legitimate integral transform of an arbitrary field with flute like symmetry has been discussed by Hazeltine and Newcomb.) If such quasimodes have a broad spectrum of $\delta_0$, then nonlinear beat waves will have a radial correlation width of $1 / k_i^2 A_\delta$, where $A_\delta$ is the width of the mode in the ballooning coordinate. Note that this same mixing length is often invoked for spectra that contain only a single $\delta_0$ (e.g., Ref. 20) in which case the mechanism for producing such a length from much broader quasimodes is unclear. Cowley has considered the toroidal mixing length problem from a different viewpoint, and shown that a single linear "twisting mode" (here termed quasimode) can encounter a secondary instability, which shortens the otherwise broad radial correlation implied by toroidal coupling. However, the study did not address the correlation length of the complicated, strongly turbulent flow that follows the onset and saturation of the secondary instability. The present description can be applied to understand the mixing length in this regime.

The results of this paper suggest several possible directions for further study. The relation between the slablike mode examined here and the ballooninglike mode of Refs. 19 and 22 should be more clearly established. It would be useful to explore the kinetic strong turbulence regime that exists between the threshold and fluid regimes (probably requiring...
particle simulations). An important line of work would be to combine the present results with those of the fluid regime to produce a prediction of $\chi$, valid for all $\eta$. A study of the basic nature of nonlinear Landau damping would be interesting, in order to understand better the result that drift waves are elastically scattered, while $\eta$ modes damp directly to the ion distribution. Finally, it would be of interest to establish the degree of generality of the formula $\chi \approx \xi^{-2}\gamma/k_i^2$ for transport from modes that saturate via nonlinear Landau damping, as the validity of such a formula could render a difficult problem more assessible.

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APPENDIX: PHASE QUANTIZATION FOR LINEAR MODES

In this appendix, the WKB approximation is used to find the solutions of Eq. (11),

$$\frac{d^2}{d\theta^2} \phi_\theta - (\mu \sin \theta) \phi_\theta = 0,$$

which vanish for $\theta \rightarrow \pm \infty$, and also the quantization condition required to connect the two vanishing solutions. Pure WKB solutions have the form

$$\phi_\pm = \exp(\pm i w),$$

where $w = |\psi_\theta| \sqrt{Q} d\theta$ and $\theta_\gamma$ is the turning point nearest to $\theta$. The complication arises because these generally will not vanish asymptotically, since such solutions acquire a component of the divergent branch at any of the infinity of turning points. Instead, the appropriate eigenstates are those for which

$$\begin{pmatrix} \phi_\alpha(	heta + 2\pi) \\ \phi_\beta(	heta + 2\pi) \end{pmatrix} = \lambda \begin{pmatrix} \phi_\alpha(\theta) \\ \phi_\beta(\theta) \end{pmatrix},$$

where $\phi_{\alpha,\beta}$ are any two linearly independent solutions of Eq. (A1). Such an eigenstate will repeat at every interval of $2\pi$, each time acquiring a factor of $\lambda$. If $|\lambda| < 1$, then $\phi$ will decrease with $\theta$, and vice versa. This corresponds to the Bloch eigenmodes, where $\phi = e^{i j f(\theta)}$, where $\alpha = \ln \lambda$ and $f$ is a function of period $2\pi$.

To find the appropriate eigenmodes with the WKB approximation, $\phi(\theta)$ is expressed as a linear combination of the dominant and subdominant solutions,

$$\phi(\theta) = \phi_\alpha(\theta) + \phi_\beta(\theta) \equiv (\Delta \theta^j, \theta_{\gamma j}^j) \phi_\alpha + (\Delta \theta^j, \theta_{\gamma j}^j) \phi_\beta, \quad (A3)$$

where $\Delta \theta \equiv \theta - \theta_{\gamma j}$ is the parallel coordinate relative to $\theta_{\gamma j}$, and $\theta_{\gamma j} = \pi j$ are the turning points of $Q = -\mu \sin \theta$. Since there are turning points at each interval of $\pi$, a given WKB basis function $(\Delta \theta^j, \theta_{\gamma j}^j)$ is valid only in the interval $\Delta \theta^j < \pi$, but after an interval of $2\pi$ it is possible to express the basis functions in terms of the $j + 2$ turning point:

$$\phi_\alpha(\theta + 2\pi) = \phi_\alpha(\theta) + \phi_\beta(\theta + 2\pi)(\Delta \theta^j, \theta_{\gamma j}^j) \phi_\beta + (\Delta \theta^j, \theta_{\gamma j}^j) \phi_\beta. \quad (A4)$$

By matching Eqs. (A3) and (A4) using the WKB connection rules, it is straightforward but tedious to show that for $j$ even (denoted by $2j$), then the dominant and subdominant basis functions will become mixed in the following way:

$$\begin{pmatrix} \phi_\alpha(\theta + 2\pi) \\ \phi_\beta(\theta + 2\pi) \end{pmatrix} = \begin{pmatrix} \cos R & \sin R \\ -\sin R & \cos R \end{pmatrix} \begin{pmatrix} \phi_\alpha(\theta) \\ \phi_\beta(\theta) \end{pmatrix} \times \begin{pmatrix} e^{-\gamma} & 0 \\ 0 & e^{\gamma} \end{pmatrix},$$

where $R = \int_{\theta_{\gamma 0}}^{\theta_{\gamma 0} + \pi} \mu \sin \theta d\theta = 2.37 \mu$. Combining Eq. (A5) with the eigenvalue equation, (A2), gives the following eigenvalues for the Bloch functions:

$$\lambda_\pm = \cos R \cosh R \pm \sqrt{\cos^2 R \cosh^2 R - 1}. \quad (A6)$$

The corresponding coefficients of the Bloch eigenvectors are given by

$$\begin{pmatrix} \phi_\alpha(\theta) \\ \phi_\beta(\theta) \end{pmatrix} = \text{const} \times \begin{pmatrix} \lambda_\pm - \cos R e^{-R} \\ \sin Re^R \end{pmatrix}. \quad (A7)$$

It is important to note that $\lambda_+ = 1/\lambda_-$, corresponding to one exponentially growing and one exponentially shrinking solution. For the case when basis functions are in terms of the odd turning points, Eq. (A5) changes slightly, and a similar analysis gives the same eigenvalues, but the corresponding eigenvectors become

$$\begin{pmatrix} \phi_\alpha(\theta) \\ \phi_\beta(\theta) \end{pmatrix} = \text{const} \times \begin{pmatrix} \sin Re^R \\ \lambda_+ - \cos Re^{-R} \end{pmatrix}. \quad (A8)$$

In the full normal mode problem of Eq. (7), there is a central potential bounded on either side by the asymptotic regions with the form of Eq. (A1). The boundaries of the central region can be defined as the two turning points $\theta_{\pm \gamma}$, beyond which the potential function $Q(\theta)$ is sufficiently close to its asymptotic limit. For simplicity, we consider only the case for which there are no turning points within the central region, and for which immediately outside $\theta_{\pm \gamma}$, are classically “forbidden” zones where $Q < 0$. (The general case is more complicated, but qualitatively not much different.) Considering the three zones (the potential well, and the potential hills immediately neighboring this), and matching derivatives and connection rules, it is straightforward to show that the central zone connects the two neighboring zones as follows:

$$\begin{pmatrix} \phi_\alpha(\theta) \\ \phi_\beta(\theta) \end{pmatrix} \begin{pmatrix} \cos S & \cos S \\ -\sin S & \sin S \end{pmatrix} \begin{pmatrix} \phi_\alpha(\theta) \\ \phi_\beta(\theta) \end{pmatrix} \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix}.$$


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where \( S = \int_{\theta}^{\theta + \pi} \sqrt{Q} \, d\theta \). The mode will vanish asymptotically if the lower (upper) solution is the negative (positive) eigenmode of Eq. (A7) [Eq. (A8)]. Putting these into Eq. (A9) shows that the condition for these two vanishing branches to be matched is \( \sin S = 0 \) or

\[
\int_{\theta_1}^{\theta_2} \sqrt{Q} \, d\theta = l\alpha,
\]

where \( l = 0, 1, \ldots \). Equation (A10) is the desired phase quantization condition. It is similar to the usual phase quantization condition connecting pure WKB basis functions, except that \( l \) replaces the usual \( l + \frac{1}{2} \). The important information gained from this analysis is that a proper normal mode is located in the central region, and nowhere else.

27. When comparing this with other theoretical spectra, one should keep in mind that \( \alpha \) has been used here, instead of the more usual \( \Sigma \). Use of the former over the latter introduces an extra factor of \( l/k \), in the spectrum, Eq. (36).
31. F. Tibone (private communication).