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Citation: Phys. Fluids B 5, 2909 (1993); doi: 10.1063/1.860679
View online: http://dx.doi.org/10.1063/1.860679
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A kinetic model of fast wave propagation in the vicinity of the minority ion cyclotron resonance in a toroidal magnetic field

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(Received 13 October 1992; accepted 19 April 1993)

Nearly all kinetic treatments of fast wave minority heating of inhomogeneous plasma in the cyclotron range of frequencies assume the magnetic field varies in the direction perpendicular to the magnetic field. However, the toroidal magnetic field of a tokamak varies along a field line due to the rotational transform and causes a small number of trapped particles to turn in the region of cyclotron resonance. In order to include the effects of rotational transform and, hence, trapped particles in the kinetic plasma response, a simplified, concentric circle flux surface model of a tokamak is employed. The most important result of this work is the derivation of response functions for Maxwellian and bi-Maxwellian minority ions which generalize and extend previous replacement Z function forms obtained from a slab approximation of a tokamak (which also retains the variation of the strength of the magnetic field along a field line). The plasma response functions obtained include both passing and trapped ions, off-axis heating, and are valid for arbitrary minority ion concentrations. The response function for a bi-Maxwellian in the case of strong anisotropy substantially modifies the Maxwellian result. Anisotropy and the effects of toroidal geometry are illustrated graphically and tend to enter at higher toroidal mode numbers. For minority concentrations of the order of less than a critical value, the plasma response functions are used to obtain the standard transmission coefficient previously obtained for straight magnetic-field models. The expression for the transmission coefficient is shown to be valid for more general unperturbed distribution functions of pitch angle and speed on each flux surface provided \( k_\parallel \rho < 1 \), where \( k_\parallel \) is the parallel wave number and \( \rho \) the minority gyroradius.

I. INTRODUCTION

Fast wave heating of tokamak plasmas in the ion cyclotron range of frequencies (ICRF) is well established in both the minority and second harmonic regimes. For both these schemes an understanding of the factors which determine the (spatial) power deposition profile is highly desirable since this would lead to greater control over the heating. Three important factors which influence the resonance width are the Doppler effect, the spatial variation of the equilibrium magnetic field (including the effects of trapped particles), and the anisotropy of the distribution function. All three effects are retained in the present treatment, which for simplicity specifically considers only the fast wave minority heating case, even though the key techniques are valid more generally.

The model which has been used in most calculations of ion cyclotron absorption in a tokamak is that of an equilibrium magnetic field with straight field lines and a perpendicular gradient in strength. This model has produced much useful information and includes the two essential ingredients of Doppler broadening and perpendicular magnetic-field inhomogeneity. However, a straight-field line model cannot account for effects due to the variation of the strength of the magnetic field along a field line.

In a tokamak magnetic field an ion (even in the zero Larmor radius approximation considered here) experiences a variation in the equilibrium magnetic-field strength due to the rotational transform. This feature has a profound effect on resonant ions compared with the straight magnetic-field model. For the latter case, an ion if resonant along a particular field line will remain in resonance since the field lines are straight. All other ions (i.e., with different values of the Larmor radius) on this particular field line will always be out of resonance. This situation also applies to the finite Larmor radius broadening, where a given ion is either in resonance over the whole of its orbit or not at all. For the toroidal model considered in this paper the situation is quite different. Since the field lines are curved, ions with different values of the Larmor radius can be in resonance on the same field line but at different spatial locations. Furthermore, these ions do not remain in resonance permanently, as in the straight-field model, but pass in and out of resonance.

Thus, for the straight-field line case there are fewer resonant ions interacting strongly and indefinitely with the wave; whereas in the curved field line model there are many more resonant ions interacting for a finite time and, therefore, more weakly with the wave.

In order to include the effect of the rotational transform, a simplified concentric circular flux surface model of the toroidal magnetic field of a tokamak is employed. Using this model means that the effect of trapped ions is included in the plasma response function. In particular, the procedure employed retains the effects of the trapped particles which turn in the resonance layer, as well as the
remaining trapped and passing particles. To date, there are very few treatments of the effect of the variation in the magnitude of the magnetic field along a field line or of trapped ions on fast wave heating at an ion cyclotron resonance. Notable exceptions are the work of Faulconer and Smith et al., in which the model employed is partially motivated by the earlier work of Itoh et al. Their plasma response function is found by retaining the leading order effect of the field inhomogeneity, as well as Doppler broadening, but without introducing the full complications of tokamak geometry.

Although the fraction of trapped ions is small they may be expected to be good absorbers, and in any case ion cyclotron absorption tends to make the ions non-Maxwellian by pushing them towards the trapped region. Grekov et al. retain the effect of trapped particles on the cyclotron absorption of fast waves at the second and higher harmonics for a bi-Maxwellian, but do not give simple, explicit results for the plasma response function. Chen and Tsai also consider trapped particle effects, but do not attempt to find a simple, explicit expression for the plasma response function. Instead, they evaluate the resonant energy absorption in the Maxwellian case using the full collisionless trajectories. The model employed in the work presented here differs from Ref. 11 by considering the minority heating regime. It also differs from Ref. 11 by implicitly assuming a collisional disruption of the particle motion along the field relative to the wave occurs between interactions with the radio frequency (rf), as described in more detail in Refs. 8 and 12–14. As a result of the collisions, only the most recent details of the particle’s trajectory need be retained in the vicinity of the resonance layer. This simplified treatment of the trajectory retains the finite wave–particle interaction time by permitting the trapped and passing particles to pass through resonance, but dramatically simplifies the mathematical description. These simplifications ultimately make it possible to model the effects of anisotropy by evaluating the plasma response function for a bi-Maxwellian unperturbed distribution function.

The plan of the paper is as follows. In Sec. II, the model of the toroidal magnetic field with circular flux surfaces is defined. The gyrokinetic equation is used to calculate the perturbed minority ion current neglecting finite Larmor radius and drift effects. In Sec. III, the resonant response function is obtained for the minority ions assuming that their equilibrium velocity distribution is Maxwellian. The response function is appropriate to trapped, as well as passing ions, is valid for arbitrary minority densities, and generalizes the results of Faulconer and Smith et al. Section IV uses this response function for a perturbative calculation of the optical depth and transmission coefficient for the fast wave crossing the minority resonance. These particular quantities are, therefore, valid only for minority densities of the order or less than a critical value given in Sec. IV. Section V generalizes the optical depth analysis further to an arbitrary minority ion distribution function and closes by evaluating the plasma response function for an arbitrary minority concentration bi-Maxwellian unperturbed distribution function.

II. PERTURBED MINORITY CURRENT DENSITY

In the vicinity of the fundamental minority resonance the cold fluid ion response is modified by kinetic effects due to Doppler shifted cyclotron damping in the inhomogeneous toroidal magnetic field. In this section, a formalism for evaluating the kinetic modifications of the perturbed minority current density in a tokamak is presented. The model considered, which neglects all finite orbit effects, is the simplest, experimentally relevant limit retaining trapped, as well as passing particle effects.

The independent spatial variables are taken to be the minor radius $r$, the poloidal angle $\beta$, and the toroidal angle $\zeta$. The major radius is $R=R_0+r\cos\beta$ with $R_0$ the location of the magnetic axis. Concentric circular magnetic flux surfaces are employed with $|\mathbf{B}|=B=\mathbf{B}_0(1-\epsilon \cos\beta), \mathbf{B} \cdot \mathbf{v}_x \approx B/R$, and $BR \approx$ constant to the requisite order, where $\epsilon=r/R_0$, in order to illustrate the procedure without introducing flux coordinates.

The velocity of a particle $\mathbf{v}$ is written as

$$\mathbf{v} = \mathbf{v}_i + \mathbf{v}_p = \mathbf{hn}_1 \mathbf{e} = (\mathbf{F} \cdot \mathbf{\phi} + \dot{\mathbf{n}}_1) \mathbf{e} + \mathbf{v}_p \mathbf{n} $$

with $\mathbf{v}_p = \dot{\mathbf{v}} \cdot \mathbf{e}$, $\mathbf{v}_i = |\mathbf{v}_i| \mathbf{e}$, $\mathbf{F} = \nabla \tau / |\nabla \tau|$ and $\phi$ the gyrophase. The independent velocity space variables $v$, $\lambda$, and $\phi$ are employed with $v=|\mathbf{v}|$ and $\lambda = \mathbf{B}_0 v^2 / 2 \mu B_0 v^2 \approx R c v^2 / R_0 c_0^2$, where $\mu = v_0^2 / 2B$ is the magnetic moment and $v_0^2 = v^2 (1-\lambda B_0)$. Using the preceding variables and neglecting finite orbit effects, the linearized gyrokinetic equation for the perturbed minority distribution function $f$ simplifies to

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\mathbf{v}_i} = \left( \frac{\partial f}{\partial q} + \frac{1}{q} \frac{\partial f}{\partial \theta} \right) - \frac{\partial f}{\partial \phi} = -S $$

(1)

with $\Omega = Ze B_0 / M c$, $S=\mathbf{a} \cdot \nabla F$, $\mathbf{a}$ the acceleration, and $F$ the unperturbed minority distribution function which is independent of gyrophase, poloidal angle and toroidal angle. For second harmonic heating the distinction between the guiding center $\mathbf{R}$ and particle $\mathbf{r}$ location must be retained in $S$ to obtain the familiar Bessel functions. Retaining the distinction between $\mathbf{R}$ and $\mathbf{r}$ would also allow non-local absorption effects due to the variation of the equilibrium magnetic field across the finite Larmor radius orbits to be treated.

Perturbed magnetic-field effects do not enter Eq. (1) when $F$ is isotropic in velocity space. For more general unperturbed distribution functions the neglect of finite gyroradius corrections in Eq. (1) also allows perturbed magnetic field effects to be ignored. In particular, for anisotropic $F$, when $k_1 v_i / \omega < 1$ with $k_1$ a typical perpendicular wave number and $\omega$ the applied rf wave frequency, the acceleration $\mathbf{a}$ may be taken to be simply

$$\mathbf{a} = (Ze/\mathbf{M}) e,$$

where $e$ is the perturbed electric field. Then $S$ may be written as

$$S = \alpha F + \Lambda \frac{\partial f}{\partial \lambda},$$

(2)
with
\[ V = a \cdot \nabla \rho, \quad A = a \cdot \nabla \rho \lambda, \]
\[ \nabla \rho \lambda = (2/\omega_0^2) [(B_0/B) - \lambda] \nabla \rho, \tag{3} \]
and \( \nabla \rho \mu = \nabla \rho \).

Considering a monochromatic wave of frequency \( \omega \) and Fourier decomposing the toroidal angle and gyrophase dependence, any linearized function \( g \) may be written as
\[ g = \Re \exp(-i\omega t) \sum_{n=-\infty}^{\infty} g_n \exp(i n \phi). \tag{4} \]
where \( \Re \) denotes that the real part is to be taken. Using the preceding on Eq. (1) gives
\[ (2\pi)^{-1} \int_0^{2\pi} d\phi \exp(-i\phi) = \frac{1}{2} \nu_1 (\bar{f} - i\bar{\mu} \times \bar{F}), \]
\[ V_{ln} = Ze v_{ln} \frac{e^*_n}{Mv} \text{ and } \Lambda_{ln} = 2B_0 \nu_0^2 V_{ln} / B^2, \]
with \( e^*_n = (\bar{f} - i\bar{\mu} \times \bar{F}) \cdot e_n \) the left-hand polarized electric-field component, gives
\[ S_{ln} = \left( \frac{Ze v_{ln} e^*_n}{Mv} \right) \left[ \frac{\partial F}{\partial \nu} + \left( \frac{2B_0 \nu_0^2}{B} \right) \frac{\partial F}{\partial \lambda} \right]. \tag{10} \]

As a result, the \( l=1 \) portion of the perturbed minority current density becomes
\[ j_{ln} = -\left( \bar{f} - i\bar{\mu} \times \bar{F} \right) \frac{(Ze)^2}{2M} \int d^2 \nu \frac{v_{ln}}{v} \int_{-\infty}^{\infty} d\tau' \left( \frac{\partial F}{\partial \nu} \right) \left( \frac{2B_0 \nu_0^2}{B} \right) \frac{\partial F}{\partial \lambda} e_n^+ (\tau') v_{ln} (\tau') \exp[i(\chi - \chi')] \right]. \tag{11} \]

Essentially the same form can be obtained for a completely general flux coordinate representation.

Because of the resonance points \( \omega = \Omega \pm n \nu_{\parallel} / R \) encountered by the minority ions during their motion, the \( \tau' \) integral cannot be evaluated by integrating by parts, as in the nonresonant or cold cases. In the next section \( j_{ln} \) will be approximately evaluated for a Maxwellian \( F \) by a technique that permits trapped, as well as passing particles to pass through and/or reflect in the cyclotron resonance region.

### III. Resonant Response Function for Maxwellian Ions

In order to obtain the resonant response function for a toroidally inhomogeneous plasma the expression for \( \chi' - \chi \) in Eq. (11) must be simplified. Assuming that collisions\(^4,12,14\) or orbit stochasticity\(^16\) disrupt the ion motion between passes through the minority resonance, then only the last pass need be retained. As a result, the most straightforward way to proceed is to approximate the rapidly varying phase factor by expanding \( \chi' \) about \( \tau \). This procedure is particularly appropriate to the present toroidal model, where the ions only interact strongly with the wave in the resonance region. This region is a narrow layer in which the ions spend only a small interval of time during their motion on a particular flux surface. Expanding \( \chi' \) to leading order in the inhomogeneity gives the result
\[ \chi' = \chi + \chi' (\tau' - \tau) + \frac{1}{2} \chi'' (\tau' - \tau)^2 + \frac{1}{3} \chi''' (\tau' - \tau)^3 + \cdots, \tag{12} \]
where \( \beta = \nu_{\parallel} / qR, \quad \Omega = \beta \mu \Omega / \beta \bar{\rho}, \quad \chi = \omega - \Omega - n \nu_{\parallel} / R, \quad \bar{\chi} = -\Omega - n \nu_{\parallel}, \quad \bar{\chi}' = -\beta \mu \Omega / \beta \bar{\rho} - \beta^3 \Omega / \beta \bar{\rho}^2 - n \nu_{\parallel} \]
with
\[ \beta = \frac{\beta^2 \Omega}{\partial \bar{\rho}} \left( \frac{1}{2 \nu_{\parallel}} \right) \frac{\lambda v R_0}{R}. \tag{13} \]
Assuming the coefficients \( e^*_n (\tau'), v_n (\tau'), v_{ln} (\tau') \), and \( B(\tau') \) are slowly varying so that they can be evaluated at \( \tau \), Eq. (11) is approximated by...
\[ j_\pi \approx -(\hat{\mathbf{r}} - \hat{\mathbf{n}} \times \hat{\mathbf{r}}) e_\pi^+ \frac{(Ze)^2}{2M} \int d^3v (v^2_\perp / v) \times \left( \frac{\partial F}{\partial v} + \frac{2Bo_\perp^2}{Bo_\parallel^3} \frac{\partial F}{\partial \lambda} \right) \int_0^\infty dt \exp\{i(\chi - \chi')\} \tag{14} \]

with \( t = \tau - \tau' \) and

\[ \chi' - \chi = - \left[ \omega - \Omega - \left( \frac{n_v}{R} \right) \right] \left( \frac{\tau}{2qR} \right) \left( \frac{v_i}{\Omega R} \right) \times \left( \frac{v_i^3}{2\Omega} \right) \left( \frac{\partial \Omega}{\partial \beta} \right)^2 \left( \frac{\tau^3}{6q^2R^2} + \cdots \right), \tag{15} \]

where \( n_v / \Omega R \ll v_i / v_i \sim e^{1/2} \) is assumed in order to neglect \( \beta \) corrections in \( \chi' \). In obtaining Eq. (15) the concentric flux surface model is employed to write \( B = 1/R \), but since \( BR \) is approximately a flux function this is not expected to be a serious limitation. If the \( \beta \) dependence of \( e_\pi^+ \) is not sufficiently slow, a Fourier decomposition in \( \beta \) must be employed to replace \( n/R \) with \( k_\parallel = (nq - m)/qR \) in Eq. (15), with \( m \) the poloidal mode number. In addition, the assumed slow variation of \( e_\pi^+ \) is related to the width of the resonance, a quantity that will emerge \( a \) posteriori. Note that the variation of \( v_i (\tau') \) with \( \tau' \) is monotonic as the particle passes through resonance so that the approximation of Eq. (15) retains those trapped ions which turn in the resonance region.

To obtain explicit results for the plasma response (reactive and dissipative) a tractable distribution function must be assumed. Taking \( F \) to be the Maxwellian

\[ F_M = \frac{(N/n_v^3/2\pi)^{3/2}}{2\pi^{3/2}} \exp(-v^2/v_i^2), \tag{16} \]

the velocity space integrals can be evaluated by employing \( \phi, v_v, \) and \( v_i \) as the variables of integration to find

\[ j_\pi \approx \frac{(Ze)^2 NR}{iM} \left( \hat{\mathbf{r}} - \hat{\mathbf{n}} \times \hat{\mathbf{r}} \right) e_\pi^+ Z(\alpha, \epsilon, n, \xi), \tag{17} \]

where \( Z(\alpha, \epsilon, n, \xi) \) is the generalized plasma dispersion function

When the poloidal dependence of \( e_\pi^+ \) is Fourier decomposed \( n \to k_\parallel R = (nq - m)/q \) in the preceding. Equation (18) is expected to be valid for \( \epsilon < 1/3 - 1/10 \) since the \( \epsilon < 1 \) assumption does not enter in a sensitive way. The parameter \( z \) characterizes the usual wave-particle, cyclotron resonant, interaction or Doppler broadening effect and \( \sigma \) arises because of Doppler modifications in the non-uniform magnetic field. The variation of the magnitude of the magnetic field along a field line is characterized by \( \alpha \) with \( \sigma \alpha \) and \( \sigma \alpha^2 \) the Doppler modifications. Effects due to the resonance occurring off the magnetic axis enter through \( \xi \) (which changes from positive on the low-field side to negative on the high) with \( \sigma \xi \) the Doppler modification.

In the fully toroidal model considered here the rotational transform is manifestly responsible for the variation of the magnetic field along a field line via Eq. (5). To remove the rotational transform it is necessary to let \( q \to \infty \) in Eqs. (18) and (19). However, the Faulconer \( ^7 \) and Smithe \( \text{et al.}^8 \) forms are obtained without explicitly introducing a rotational transform since their replacement \( Z \) function depends on a magnetic-field scale length \( L_\parallel \). By identifying their \( L_\parallel \) as \( qR/e \) and suppressing collisions all the descriptions become identical for an on-axis resonance and \( \sigma \to 0 \). By using the substitution \( n/R \to k_\parallel \) the asymmetry in the \( n \to k_\parallel /R \) spectrum noted in Ref. 8 is recovered.

When the terms involving \( \sigma \) are sufficiently small (\( \sigma < |\alpha| < 1, |\sigma| |\xi| < |\alpha|^3 \) or \( \sigma > 1, \sigma(\alpha^2 + |\xi|^2) < |\alpha|^{1/2} \) ), Eq. (18) reduces to the result of Faulconer \( ^7 \) and the collisionless form of the replacement plasma response function of Smithe \( \text{et al.}^8 \) namely

\[ Z(\alpha, \epsilon) = i \int_0^\infty dx \exp \left( ix - \left( 1 - \frac{1}{2} \alpha x \right)^2 \left( x^2/4 \right) \right), \tag{20a} \]

which they obtain heuristically by employing an ingenious set of simplified trajectories that conserve both energy and magnetic moment and are motivated by the earlier work of Itoh \( \text{et al.}^{12} \) which noted the important role of the inhomogeneity as \( n \to 0 \). As \( n \to 0, |\alpha|^{1/2} \geq 1 \) and Eq. (18) simplifies to
FIG. 1. The real (solid line) and imaginary (broken line) parts of Z(\(z, \alpha, \sigma, \xi\)) plotted as a function of \(z\) for \(\alpha=100\) and \(\sigma=0=\xi\), corresponding to an on-axis minority resonance. The width of the resonance layer is given by \(|z| \sim |\alpha|^{1/2}\) to be \(\Delta R\sim (r/q)^{1/2}\), where \(\rho=\nu_T/\omega\).

\[
Z(z,\alpha,\sigma,\xi) \big|_{n \to 0} = \left( \frac{2i}{|\alpha|^{1/2}} \right) \int_0^\infty \frac{dy \exp(\text{i}y^4/[1+i(\kappa+\gamma)y^3])}{(1-\text{i}ky^3)^2[1+i(\kappa+\gamma)y^3]^{1/2}}.
\]

(20b)

where \(y=\frac{1}{2}|\alpha|^{1/2}s, s=2z/|\alpha|^{1/2}, \kappa=8s|\alpha|^{1/2}/3,\) and \(\gamma=8s\xi/3|\alpha|^{3/2}\) are independent of \(n\). Since \(\kappa\sim (r\rho)^{1/2}/R<1\) and \(\gamma\sim (\rho/r)^{1/2}<1\), the new terms give only small corrections for \(|\alpha|^{1/2}>1\) to the results of Refs. 7 and 8, where \(\rho=\nu_T/\omega\).

When \(|\alpha|\) is sufficiently large the new terms are negligible, the inhomogeneity dominates, and the strongest wave-particle interactions are near the cold minority resonance for both signs of \(\alpha\), as can be seen from the \(n\to 0\) form of Eq. (20b). For \(|\alpha|>1\), the power deposition layer broadens\(^6\) to \(|z| \sim |\alpha|^{1/2}\) from \(|z| \sim 1\) and the amplitude of \(Z\) is reduced\(^6\) by \(|\alpha|^{1/2}\), as shown in Fig. 1. This case is expected to be of particular interest for small \(|n|\) operation.

When the inhomogeneity is weak it is tempting to neglect the \(\alpha\) dependence of Eq. (20) and simply replace it by the usual \(Z\) function. However, Faulconer\(^7\) and Smithe \textit{et al.}\(^8\) have shown that for \(\alpha>0\) there are important corrections. For \(1>\alpha>0\) these contributions to Eqs. (18) and (20) arise from the region \(x\approx 2/\alpha\) and lead to oscillatory behavior about \(z=0\). Evaluating the contribution to Eq. (18) from \(x\approx 2/\alpha\) by a saddle point method for \(0<\alpha|z|<1,\) \(\sigma<\alpha<1,\) and \(|\xi|\leq \sigma^2\) results in

\[
Z(z,\alpha,\sigma,\xi) = Z(z) + i2\pi^{1/2} \left( 1 - \frac{i4\sigma}{3\alpha} \right)^{-2} \left[ \begin{array}{c} 1 \\ \frac{i4\sigma}{3\alpha} \left( 1 - \frac{2\xi}{\alpha^2} \right)^{-1} \end{array} \right] \times \exp \left[ \frac{i2z}{\alpha} - \frac{z^2}{2} \left( 1 - \frac{i4\sigma}{3\alpha} \left( 1 - \frac{2\xi}{\alpha^2} \right) \right) \right],
\]

(21)

where \(Z(z)\) is the usual \(Z\) function (and the saddle at \(x\sim 1/\alpha\) gives a negligible contribution as long as \(\alpha\) is sufficiently small). The contribution from the saddle at \(x\sim \gamma/\alpha\) arises because the particle motion along the twisting field line causes it to pass through an effective resonance region, where the field inhomogeneity modification cancels the Doppler shift [at the time at which the two terms linear in \(v_{\parallel}\) cancel in Eq. (15)]. For an asymmetric spectrum this causes an upper-down asymmetry in the wave propagation and absorption. As \(\alpha\to 0\), collisional disruption of a particle’s streaming motion\(^12\) removes the oscillatory behavior in the plasma response function\(^8\) (different collisional mechanisms are considered in each of these references).

The fine-scale oscillations introduced by \(\exp(\text{i}2z/\alpha)\) for \(|z|<1\) as \(\alpha\to 0\) are collisionally modified by the new terms when \(\sigma=\alpha\) or \(\sigma|\xi|\leq \alpha^2\); in most cases making it unnecessary to retain collisions. For still smaller \(\partial n/\partial \beta\) and/or larger \(|n|\) such that \(0<\alpha|\sigma|<\alpha^2\|\sigma|\) the amplitude of the fine-scale oscillations becomes vanishingly small with only \(Z(z)\) surviving in Eq. (21) as \(\alpha\to 0\), as shown in the figures. Figures 2(a), 2(c) plot the real (solid line) and imaginary (broken line) parts of Eq. (18) for the minority resonance at the magnetic axis (\(\theta/\Omega=0\)) and illustrate the effect of finite \(\sigma|\xi|\) which requires \(n^3 \geq r^2/RQ\). A large value of \(\sigma=0.1\) is used since the \(\alpha=0.1\) oscillations can be more conveniently displayed than those arising for \(\sigma=0.01=\sigma\). For Figs. 3(a)–3(b), the minority resonance is shifted off axis by \(\alpha\|\xi\) to show \(n_\| \geq r/Q\) results. The plot for \(\alpha=0.1, \sigma=0.01,\) and \(\xi=0\) is quite similar to Fig. 2(a). For Fig. 3(b), \(\xi\) is so large that \(\alpha\) no longer plays a role and can be set to zero to obtain the same plot. Positive values of \(\xi\) result in similar plots but with more of the structure at \(z>0\), rather than at \(z<0\).

In the next section, the cold fluid theory of fast wave minority heating is modified by making the substitution

\[
\frac{1}{\omega-\Omega} = \frac{-RZ(z,\alpha,\sigma,\xi)}{|n|v_T},
\]

(22)

(Notice that for \(|z|>1, Z \sim -1/z\). This procedure allows the optical depth and transmission coefficient for fast wave minority heating to be evaluated analytically by a perturbation expansion in the minority concentration.)
IV. OPTICAL DEPTH AND TRANSMISSION COEFFICIENT FOR A MAXWELLIAN $F$

The fast (or compressional Alfven) wave in a cold fluid description of minority heating satisfies the dispersion relation,

$$n_1^2 = (\epsilon_1 + \epsilon_x - n_0^2)(\epsilon_1 - \epsilon_x - n_0^2)/(\epsilon_1 - n_0^2) \quad (23a)$$

with $n_0 = k_0 c/\omega$, $n_1 = k_1 c/\omega$,

$$\epsilon_1 = 1 - \sum_j \omega_p^2_j / (\omega^2 - \Omega_i^2)$$

and

$$\epsilon_x = \sum_j \omega_p^2_j \Omega_i / (\omega^2 - \Omega_i^2).$$

For a plasma with a majority ion species ($j=i$) and a single minority species (unsubscripted), the replacement (22) in minority terms proportional to $1/(\omega - \Omega)$ yields

$$n_1^2 = (\epsilon_1 + \epsilon_x - n_0^2)(\epsilon_1 - \epsilon_x - n_0^2)/(\epsilon_1 - n_0^2) \quad (23a)$$

and

$$\epsilon_1 \approx -\omega_p^2 \left( 1 - \frac{\eta R Z(z, \alpha, \sigma, \xi)}{2\omega |n| v_T} \right)$$

For $\sigma - \alpha \approx |n|^2 r R / q f^2$ and (c) $\alpha = 0.01$ and $\sigma = 0.1$, where

FIG. 2. Real (solid) and imaginary (broken) parts of $Z(z, \alpha, \sigma, \xi)$ as a function of $z$ for an on-axis resonance, $\xi=0$: (a) $\alpha=0.1$ and $\sigma=0$, where the period of the oscillations is given by $|t| - u_0$; (b) $\alpha=0.1$ and $\sigma=0$, where $\sigma \approx 0$ or $\sigma \gg r R / q f^2$.

To evaluate the optical depth $\kappa$ of the minority resonance and, therefore, the transmission coefficient

$$T = \exp(-\kappa).$$

$\kappa \approx 1$ is required, where $\text{Im}$ and $\text{Re}$ denote the imaginary and real parts. The transmission coefficient $T$ is a measure of how much energy the incident wave loses in crossing the interaction region. The evaluation of $T$
so that

\[ n_1 - n^* = \frac{\eta \omega_p^2 R (\bar{e}_1 + e_1 - n^2_1)^2 [Z(z,\alpha,\sigma,\xi) - Z^*(z,\alpha,\sigma,\xi)]}{4\omega |n| \nu l \bar{n}_1 (e_1 - n^2_1)^2} \] (25)

to the requisite order, where a superscript asterisk denotes complex conjugate.

When the scale lengths of the variations of the electric field \( e_0 \) are long compared to the width of the minority resonance the optical depth

\[ \kappa = 2 \int dR \text{Im} k_1 \left| \frac{R}{\omega} \right| \int dR \left( n_1 - n^*_1 \right) \] (26)

is only sensitive to the \( R \) dependence of \( z = R(\omega - \Omega)/|n| \nu_1 \). Using \( dR \approx (|n| \nu_1/\omega) dz \), Eqs. (25) and (26) yield

\[ \kappa \approx \frac{\eta \omega_p^2 R (\bar{e}_1 + e_1 - n^2_1)^2}{4\omega |n| \nu_1 (e_1 - n^2_1)^2} \left| \int_{-\infty}^{\infty} dz [Z(z,\alpha,\sigma,\xi) - Z^*(z,\alpha,\sigma,\xi)] \right|, \] (27)

where all spatially varying quantities outside the \( z \) integral are slowly varying and are to be evaluated at the minority resonance as indicated. Noting that the only sensitive spatial variation in \( Z \) is via the \( \exp(izx) \) and gives

\[ \int_{-\infty}^{\infty} dz [Z(z,\alpha,\sigma,\xi) - Z^*(z,\alpha,\sigma,\xi)] = 4\pi |x\delta(x)| = 2\pi i \] (28)

so that the optical depth becomes

\[ \kappa \approx \frac{\pi \eta \omega_p^2 R (\bar{e}_1 + e_1 - n^2_1)^2}{2\nu_1 (e_1 - n^2_1)} \] (29)

which for small \( n_1 \) reduces to

\[ \kappa \approx \frac{\pi \eta \omega_p^2 R (\omega - \Omega_l)^2}{2\cos \Omega_l} \] (30)

since \( \bar{n}_1 \to \omega_p/\Omega_l \). Equations (29) and (30) agree with the results of Francis, Bers, and Ram \(^8\) and with Lashmore-Davies and Dendy \(^9\) if \( L_p \) is identified as \( R \). When \( n_1 \ll k_1 = (nq - m)/qR \) is employed in Eq. (29) there is an asymmetric spectrum shift in \( T \) due to the poloidal field modification \((m/qR)\) of \( n/R \) in essentially the same way as found in the numerical results of Smithe et al. \(^8\)

The preceding evaluation provides yet another demonstration of the robustness in the optical depth and transmission coefficient when \( \eta \) is small since the result is independent of the details of the absorption process and, therefore, Doppler broadening \((n \text{ or } k_1 \text{ in } Z)\) and inhomogeneity effects \((\alpha \text{ and } \xi \text{ in } Z)\). This robustness is investigated further in the next section, where \( \kappa \) and \( T \) are evaluated for more general equilibrium distribution func-

FIG. 3. Real (solid) and imaginary (broken) parts of \( Z(z,\alpha,\sigma,\xi) \) as a function of \( z \) for an off-axis resonance, \( \xi \neq 0 \), with \( \alpha = 0.1 \) and \( \sigma = 0.01 \): (a) \( \xi = -1 \), where \( \sigma |\xi| \gg \alpha^2 \) or \( |\alpha| \gg \sigma \) and (b) \( \xi = -100 \), where \( \sigma |\xi| \ll 1 \) gives \( |\alpha| \approx \sigma \) or \( |\alpha| \approx \sigma \).

gives no information on reflection, absorption, or mode conversion; all of which must be obtained from a full wave description.

Taylor expanding \( n_1^2 \) about \( \eta = 0 \) and denoting quantities evaluated at \( \eta = 0 \) by an overbar gives

\[ n_1^2 = \bar{n}_1^2 + \frac{\eta \omega_p^2 R Z(z,\alpha,\sigma,\xi) (e_1 + e_1 - n_1^2)^2}{2\omega |n| \nu l (e_1 - n_1^2)^2} + \cdots \]
tions, but does not imply that the plasma response function and, hence, the power deposition profile display the same insensitivity.

Before concluding this section we observe that the formula for the optical depth given by Eq. (29) was obtained by an expansion in the minority to majority density ratio. This expansion will be valid as long as the minority resonance is indistinguishable from the hybrid resonance. However, as the minority to majority density ratio increases, these two resonances begin to separate and when they are well separated the quantity $\epsilon_1 - n_i^3$, which occurs in the denominator of Eq. (25), can vanish. In this region, the expansion clearly breaks down. One might still apply the previous analysis at the minority resonance but now important phenomena occur in the region of the hybrid resonance, namely reflection of the incident fast wave for a low-field side antenna and mode conversion to the ion Bernstein wave for high-field side incidence. In general, the larger the separation of the minority and ion-ion hybrid resonances, the weaker the damping in the vicinity of the minority resonance.

An important quantity for fast wave minority heating is the minority to majority density ratio for which the hybrid and minority resonances are degenerate. This critical ratio is also close to the condition for maximum minority absorption. As the minority to majority density ratio is increased beyond this critical value the two resonances move apart until they become quite distinct. The critical density ratio can be calculated for the present toroidal model using the argument given in Ref. 21. For small values of $n_i$ (i.e., $a_\rho \ll c k\|$) the condition for hybrid resonance is $\Re (\epsilon_1) = 0$. Degeneracy of the minority and ion-ion hybrid resonances occurs when $\Re Z(x, \alpha, \sigma, \xi)$ takes its maximum value at the position where $\Re (\epsilon_1) = 0$. Assuming $\omega = \Omega$, the equation $\Re (\epsilon_1) = 0$ can be solved for the critical density ratio by inserting the maximum value of $\Re Z(x, \alpha, \sigma, \xi)$ giving

$$N_{\text{crit}} = \frac{2 M_1 |n_i| v_i}{N_i} \left( \frac{2 M_1 |n_i| v_i}{N_i} \right)^2 - 1 \left( \frac{\Re Z}{\max} \right).$$

(31)

We note that for a straight field line model $|n_i|/R$ is identified as $k_\|$ and Eq. (31) reduces to Eq. (37) of Ref. 21, since the plasma response function becomes the plasma dispersion function in this limit. However, the toroidal calculation is more general and contains additional resonance broadening terms. Thus, referring to Fig. 1, where $\alpha = 100$, we see that $\Re Z_{\max} = 0.13$ compared with unity for the plasma dispersion function. This means that the resonance is much broader for this case and, hence, the critical density ratio is much larger thus allowing strong damping for a wider range of minority densities. We also note that the damping remains even in the limit $k_\| = 0$, as already pointed out by Smith et al.\(^8\) Of course, in the toroidal model, $k_\|$ cannot be zero over the whole resonance region but only at a point in the region.

V. NON-MAXWELLIAN DISTRIBUTION FUNCTIONS

The evaluation of the optical depth (or transmission coefficient) is generalizable to arbitrary isotropic and anisotropic distribution functions $F$ that need not be even functions of parallel velocity (but are still independent of $\phi$, $\beta$, and $\xi$). To this end a further generalization of the plasma response function, $\hat{Z}$, is defined by comparing Eqs. (11), (14), and (17) to find

$$\hat{Z} = \frac{-i}{2 N R} \int d^3v \frac{v_i}{v} \left( \frac{\partial F}{\partial v} + \frac{2 B_0 v_i^2}{B_0^2} \frac{\partial F}{\partial \lambda} \right)$$

$$\times \int_{-\infty}^{\infty} d^3v^\prime \exp[i(\chi - \chi^\prime)],$$

(32)

which reduces to Eq. (18) when Eq. (15) is employed and $F$ is Maxwellian.

Once $\chi^\prime$ is Taylor expanded about $\tau$ as in Eqs. (12) or (15) and it is observed that the only rapid spatial variation is in $\chi = \omega - \Omega - n v_i / R$, then proceeding as in Eq. (28) and letting $\tau = \tau - \tau^\prime$ gives

$$\left( \frac{\omega}{v_i} \right)^2 \int_{-\infty}^{\infty} d\tau^\prime \exp[i(\chi - \chi^\prime)]$$

$$\approx 2 \pi \int_{0}^{\infty} d\tau \delta(1) = \pi,$$

(33)

where $\omega = [R(\omega - \Omega) - n v_i] / n_i$ and $dz = [1 - (n \lambda w v_i/2 R^2)](\omega dR / |n_i| v_i)$. Therefore, a general expression for the optical depth is

$$\kappa = \bar{\kappa} \gamma$$

(34)

with $\bar{\kappa}$ defined in Eq. (29) and

$$\gamma = \left( \frac{1}{2N} \right) \int d^3v v_1 \frac{\partial F}{\partial v_1} |_{\Omega = \omega - \eta_i / R}$$

(35)

provided $\nu \ll \omega R \ll 1$. In carrying out the preceding steps

$$(1/\nu) \left[ \frac{\partial F}{\partial v} + \frac{2 B_0 v_i^2}{B_0^2} \frac{\partial F}{\partial \lambda} \right] = \frac{1}{v_i} \frac{\partial F}{\partial v_1}$$

(36a)

is employed. Then using $d^3v = 2 \pi v_1 dv_1 dv_1$ and

$$\int_{0}^{\infty} dv_1 v_1^2 \frac{\partial F}{\partial v_1} = -2 \int_{0}^{\infty} dv_1 v_1 F,$$

(36b)

Eq. (35) becomes

$$\gamma = (1/N) \int d^3v F |_{\Omega = \omega - \eta_i / R} = 1,$$

(37)

where the last step follows because $F$ must be a flux function and all other poloidal angle dependences are slow. As a result of the preceding argument, the optical depth (or transmission coefficient) is insensitive to the details of the minority distribution function, as well as Doppler and inhomogeneity effects, for minority concentrations less than the critical value given by Eq. (31). Only the quantities entering $\bar{\kappa}$ (or $\kappa_0$ if $k_\| = 0$) matter. Therefore, under such conditions the optical depth is only sensitive to the cold
fluid minority response, since replacing \( Z \) by its cold approximation gives the only contribution that survives, namely,

\[
\int_{-\infty}^{\infty} dz \, Z \rightarrow \int_{-\infty}^{\infty} \frac{d\Omega}{(\omega - \Omega)} = n_i. \tag{38}
\]

The insensitivity of \( \gamma \) to Doppler broadening, inhomogeneity effects, and the speed and pitch angle dependence of the unperturbed distribution function is quite striking, but does not imply that the plasma response function \( \tilde{Z} \) of Eq. (32) displays the same insensitivity. Inserting Eq. (36a) into Eq. (32) gives

\[
\tilde{Z} = -i\frac{n_i v_i}{2NR} \int d^3v \frac{\partial F}{\partial v_i} \int_{-\infty}^{\infty} d\tau' \exp[i(\chi - \chi')] \tag{39}
\]

Under the conditions noted prior to Eq. (20a) the \( \gamma' \) and \( nt^2 \) terms can be neglected in Eq. (15) to obtain

\[
\chi - \chi' = \left( \omega - \Omega - \frac{n v_i}{R} \right) + \left( \frac{v_i^2}{2gR} \right) \partial \Omega \partial \beta. \tag{40}
\]

In this simplified limit, Eq. (36b) may be used to express Eq. (39), with \( \chi - \chi' \) given by Eq. (40), as

\[
\tilde{Z} = -i\frac{n_i v_i}{NR} \int d^3vF \int_{-\infty}^{\infty} dt \exp[i(\chi - \chi')]. \tag{41}
\]

From this last form it is seen that \( \tilde{Z} \) is only sensitive to the \( v_i \) dependence of \( F \) in this simplified limit, making the plasma response function \( \tilde{Z} \) more sensitive to Doppler and inhomogeneity effects and the details of the unperturbed distribution function than the optical depth and transmission coefficient. If the \( v_i \) dependence of the distribution function \( F \) (which must be a function of \( v, \lambda, \) and \( \epsilon \) only) is Maxwellian in \( v_i \) then Eq. (41) reduces to Eq. (20a) for an arbitrary \( v_i \) dependence.

In Eq. (48) the only new parameter due to the anisotropy is

\[
k = \frac{A\partial \Omega}{4\eta n \Omega}, \tag{49}
\]

which carries the same sign as \( \alpha \) and enters to enhance the role of the perpendicular portion of the distribution function and, therefore, the trapped particles, via the \( k\alpha x^2 \) and \( kv^2 \) terms.

Superficially, Eq. (48) is very much the same as Eq. (18); however, \( \alpha/2 \) has been replaced by \( k \) in one term in the denominator and \( 2k/\alpha \) for many high power minority heating experiments. For very large \( A \) the \( \sigma \) terms may be safely neglected in Eq. (48) to obtain the less complicated form

\[
\tilde{Z}_b(x, \alpha, k) = i \int_{0}^{\infty} dx \exp \left[ i\pi - \frac{(1 - \frac{1}{2}k x)^2 x^2}{4[1 - i(\alpha - \frac{1}{2}x^2 - \frac{1}{2}\alpha x^2)]} \left[ 1 + i(1 - \frac{1}{2}\alpha x)k x^2 \right] \right]^2. \tag{50}
\]

In order to obtain a plasma response function more general than Eq. (20a) it is necessary to employ Eq. (39) with the full expression for \( \chi - \chi' \), as given by Eq. (15). Consider, for example, the bi-Maxwellian

\[
F = \eta_0 \exp(-\beta_1 v^2 - \alpha || v^2), \tag{42}
\]

where \( w \) and \( u \) are the perpendicular and parallel velocities evaluated at \( \beta = 0 \), namely,

\[
w = v(\lambda - \epsilon \lambda)^{1/2} \quad \text{and} \quad u = v(1 - \lambda + \epsilon \lambda)^{1/2}, \tag{43a}
\]

and \( \beta_1, \alpha ||, \) and \( \eta_0 \) must be flux functions. Using \( u^2 = v^2 - w^2, v^2 = v_i^2 + v_p^2, \) and \( v_i^2 = BV^2 / B_0 \) gives

\[
w^2 = (1 - \epsilon)(B_0 / B)v_i^2 \quad \text{and}
\]

\[
u_2 = u_i^2 + [1 - (1 - \epsilon)(B_0 / B)]v_i^2. \tag{43b}
\]

As a result, \( F \) may be rewritten as

\[
F = \eta_0 \exp(-\alpha_1 v_i^2 - \alpha || v_p^2) \tag{44}
\]

with

\[
\alpha_1 = \beta_1 (1 - \epsilon)(B_0 / B) + \alpha || [1 - (1 - \epsilon)(B_0 / B)], \tag{45}
\]

Evaluating the density \( N = \int d^3v F \) by integrating over \( v_i \) and \( v_p \) gives

\[
\eta_0 = n^{-3/2} \alpha_1 \alpha ||^2 N, \tag{46}
\]

where \( \alpha_1, N \) must be a flux function.

Next, the plasma response function of Eq. (39) is evaluated by inserting Eqs. (15), (44), and (46) and carrying out the \( v_i \) and \( v_p \) integrations, just as in the Maxwellian case. The only new feature that arises is in the \( v_i \) integral because \( \alpha_1 \neq \alpha || \). To characterize the anisotropy let

\[
\alpha || = 1/\eta_i \quad \text{and} \quad \alpha || = 1/\eta_0 \tag{47}
\]

where the anisotropy factor \( A \) is approximately the ratio of the perpendicular over the parallel energy content of the bi-Maxwellian. Then, the plasma response function for the bi-Maxwellian of Eq. (44) is

\[
Z_b(z, \alpha, \sigma, \xi, k) = i \int_{0}^{\infty} dx \exp \left[ i\pi - (1 - \frac{1}{2}k x)^2 x^2 \right] \frac{1}{4[1 - i(\alpha - \frac{1}{2}x^2 - \frac{1}{2}\alpha x^2)]} \left[ 1 + i(1 - \frac{1}{2}\alpha x)k x^2 \right] \left[ 1 + i(1 - \frac{1}{2}\alpha x)k x^2 \right]^2. \tag{50}
\]
Equation (48) includes the effects of an anisotropy in the distribution function, the rotational transform of the magnetic field, trapped and passing particles, and Doppler broadening. The substitution noted in Eq. (22) allows Eq. (48) to be used to generalize cold plasma models of fast wave heating in tokamaks to include these kinetic effects on the minority species.

VI. DISCUSSION

In the preceding sections the plasma response function is derived for fast wave minority heating from a fully toroidal description that retains the inhomogeneity of the magnetic field, including trapped particle effects, as well as the more familiar Doppler effect. Equation (18) is the general result for a Maxwellian, while Eq. (48) is a still more general expression since it is valid for a bi-Maxwellian distribution function. Both expressions retain trapped particles, including those turning in the resonance layer; however, the integrations over all velocity space make it difficult to identify specific effects associated with the trapped population. Roughly speaking, the terms $\alpha x^3$ and $k x^4$, which come from the $t^5$ terms in Eq. (15), are expected to be most strongly influenced by the trapped particles. In particular, recall that Eqs. (18) and (48) differ expected to be most strongly influenced by the trapped particles. To identify clearly the role of the trapped fraction would require a far more realistic anisotropic unperturbed distribution function $F$ having the appropriate pronounced velocity space structure due to the trapped particles that turn in the vicinity of the minority resonance. We have not yet found such an equilibrium distribution function which is sufficiently simple to enable an analytically tractable plasma response function to be obtained. Other potentially important effects that are neglected include finite Larmor radius and banana width effects and a rigorous treatment of pitch angle scattering collisional modifications.

In the absence of toroidal effects the resonance or power deposition layer width is given by $|z| = |(\omega - \Omega) R/n v_t| \sim 1$. For on-axis heating $\Omega R = \omega R_0$ may be inserted to obtain the width

$$\frac{|R - R_0|}{R} \sim \frac{|n v_T|}{\omega R} \approx k \parallel \rho.$$ 

When toroidal effects are retained for the Maxwellian case additional scale lengths enter. For $0 < \alpha \sim n' \approx q n^2 < 1$ oscillations are introduced in the plasma response function with a scale of $|z| \sim \alpha$ or

$$\frac{|R - R_0|}{R} \sim \frac{\epsilon}{q |n|},$$

as can be seen from Eq. (21). These oscillations become less pronounced when $n' \approx R/\rho q$ (or $\alpha$) for on-axis heating or $n' \approx R/\rho q$ ($\sigma |\xi| \approx \alpha^2$) for off-axis heating. For monopole operation or for parameters such that toroidicity dominates ($|\alpha|^{1/2} \gg 1$), Eq. (20a) or Eq. (20b) gives the extended power deposition width from $|z/\alpha^{1/2}| \sim 1$ to be
since $\sigma$ and $\xi$ corrections are small.

For the bi-Maxwellian case with $1 \gg \alpha > 0$ the power deposition width remains at $|R - R_0| \sim k_R \rho R$ and the amplitude of the $|z| \sim \alpha$ oscillations in $Z_b$ starts to decrease when $|k| \sim \alpha^2$ or $A \approx \alpha \rho \nu \rho^2$. The small $\alpha$ oscillations disappear by $k \sim 1$ ($A \approx \rho \nu \rho^2$) when the power deposition width begins to broaden to $|z| \sim k^{1/2}$. As $\alpha$ is increased to $k \sim 1$ the oscillations return but only for $z < 0$. For large anisotropies such that $|k| \gg |\alpha|^{1/2}$ the $z < 0$ oscillations become much more extended than the power deposition width which is given by $|z/k^{1/2}| \sim 1$ to be

$$\frac{|R - R_0|}{R} \sim \left(\frac{\epsilon \rho}{q R}\right)^{1/2},$$

When $|k| \gg |\alpha|^{1/2} > 1$, $z < 0$ oscillations are much less rapid and not as extended and the power deposition width is found from $|z| \sim |ak|^{1/3}$ for this strong anisotropy case to be given by

$$\frac{|R - R_0|}{R} \sim \left(\frac{A \rho \nu^2}{q^2 R^4}\right)^{1/3}.$$

Recall that to remove the electric field from under the trajectory integral in Eq. (11) it was necessary to assume that it varied on a scale much longer than the power deposition width. The preceding estimates indicate that the deposition width varies between $|n| \rho$ and $(\rho \nu \rho^2)^{1/2}$ for a Maxwellian, and over a similar range for a bi-Maxwellian unless it is highly anisotropic [in which case the width can become either $(A \epsilon |n|/q)^{1/2} \rho$ or $(A \epsilon \rho^2/q^4 R)^{1/2}$]. A full wave treatment is required to verify a posteriori that the electric field varies on a much longer scale length.
The preceding discussion of the behavior of the plasma response function and the results of Secs. III and V indicate that the toroidicity of the magnetic field and the anisotropy of the unperturbed distribution function will complicate substantially a full wave evaluation of the transmission, reflection, and absorption/mode conversion coefficients of the fast wave in a minority heated tokamak during either monopole (small \(|n|\)) or dipole (large \(|n|\)) operation. For the Maxwellian case the inhomogeneity effect shown in Eq. (20a) and first found in Refs. 7 and 8 will be more important for low \(|n|\) operation than for large \(|n|\). The new inhomogeneity terms from Eq. (18) that do not appear in Eq. (20a) are more important in dipole operation, where the larger \(|n|\) are more likely to make \(\sigma \gtrsim \alpha\) or \(\sigma \gtrsim \alpha^2\). However, for off-axis heating with the resonance nearly tangent to a flux surface (\(\sigma \sim 0\)), the \(\sigma^2\) term tends to be more important for monopole operation. Anisotropic effects also tend to be more likely in dipole operation since for \(\alpha < 1\) only \(k \gtrsim \alpha^2\) need be satisfied. However, for very strong anisotropies, \(|k| \sim 1\sim |\alpha|\) and \(|k| \gtrsim |\alpha|^{-1/2}\) are more easily satisfied in monopole operation. In addition, \(\alpha\) and \(k\) vary substantially along cylindrical surfaces of constant \(R\). More realistic unperturbed distribution functions are not yet tractable, but are certain to introduce further complications since they cannot be characterized by two temperatures.

We emphasise that the resonant plasma response functions calculated in this paper are valid for arbitrary minority ion densities. On the other hand, the results obtained in Secs. IV and V, demonstrating the insensitivity of the optical depth and transmission coefficients to inhomogeneity and anisotropy, are only valid for minority to majority density ratios less than or of the order of the critical ratio given by Eq. (31). For larger ratios the ion-ion hybrid resonance occurs for which a full wave theory is required. The fully toroidal, kinetic evaluation of the plasma response function for the resonant minority ions is an essential step towards the construction of both one and two-dimensional full wave theories for fast wave minority heating.

ACKNOWLEDGMENTS

P.J.C. is grateful to David Start and Miro Bures of the Joint European Torus for beneficial discussions on minority heating and to Jim Myra of Lodestar for his many insights as well as his comments on the original manuscript. Special thanks are due to Pat Colestock of Fermi Lab for making available notes for the work published in Ref. 8.

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