Magnetohydrodynamic stability at a separatrix. II. Determination by new conformal map technique

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It was shown in the first part to this paper how a simple magnetohydrodynamic model can be used to determine the stability of a tokamak plasma’s edge to peeling (external kink) modes. It was found that stability is determined by the value of $\Delta'$, a normalized measure of the discontinuity in the radial derivative of the radial perturbation to the magnetic field at the plasma-vacuum interface. To avoid the possibility that numerical divergences near the $X$-point might lead to misleading conclusions about plasma stability, this paper calculates the value of $\Delta'$ analytically. This is accomplished by showing that the method of conformal transformations can be applied to systems with a continuously varying nonzero boundary condition and using the technique to obtain analytical expressions for both the vacuum energy and $\Delta'$. A conformal transformation is also used to obtain an equilibrium vacuum field surrounding a plasma with a separatrix and $X$-point. This allows the analytical expressions for the vacuum energy and $\Delta'$ to be evaluated. The results here, combined with those in the first part of this paper, subsequently provide a quantitative description of the peeling mode’s growth rate as the plasma-vacuum boundary more closely approximates a separatrix with an $X$-point. [DOI: 10.1063/1.3194271]

I. INTRODUCTION

The first part of this paper generalized a simple model for peeling modes (PMs), from cylindrical to toroidal geometry, and found that PM stability was determined by the value of a single parameter $\Delta'$. This paper calculates $\Delta'$ analytically, so as to avoid the divergences that can arise in numerical calculations near an $X$-point.

The outline of the paper is as follows. Conformal transformations are reviewed in Sec. II and the Karman–Trefftz transformation1 is described in Sec. III. The Karman–Trefftz transformation provides an example of a transformation from a circular boundary to a separatrix boundary with an $X$-point. Section IV reviews how a complex potential may be defined and used to calculate how the vacuum magnetic field will transform under a mapping from a system with a circular boundary, to one with a separatrix. For a large aspect ratio system, the vacuum energy is calculated for both a circular boundary and a separatrix cross section in Sec. V, obtaining the vacuum energy for a separatrix cross section in terms of a sum of Fourier coefficients. The Fourier coefficients are determined by the plasma-vacuum boundary conditions, and Sec. VI discusses how these boundary conditions are modified by a conformal transformation. This is where we have departed from the conventional textbook applications of conformal transformations that require the boundary conditions on the function’s normal derivative to be zero. Instead the transformed boundary conditions presented in Sec. VI provide analytic expressions to determine the Fourier coefficients in terms of the straight field line angle that is not yet known. The straight field line angle is calculated in terms of an equilibrium vacuum field using the conformal transformation technique in Sec. VII. Section VIII calculates an equilibrium vacuum field for both the circular boundary and the separatrix boundary systems, subsequently allowing analytic expressions for the safety factor $q$ and the straight field line angle to be obtained at the plasma-vacuum boundary. At this point all the analytic expressions needed to calculate the vacuum energy have been obtained. Sections IX–XI analytically calculate $\Delta'$ and the vacuum energy $\delta W_v$. These calculations are described in detail in the supplementary material,2 the calculations are summarized and outlined here. Section XII compares and extends previous work. Section XIII summarizes the paper.

II. CONFORMAL TRANSFORMATIONS

A conformal transformation $w(z)$ (e.g. see Ref. 3), is an analytic function between complex planes $z \rightarrow w(z)$. It has the property that the angle (and direction of the angle), between curves in the plane from which they are mapped, is retained between the resulting curves in the plane onto which it is mapped. This angle-preserving property ensures that the unit normal to a line in one plane will map to a vector normal to the mapped line, with a consequence that a boundary condition for a vector field with zero normal component, will be transformed to a boundary condition in which the vector field’s normal component is zero also. More generally, Sec. VI shows how the boundary condition is modified when a vector field’s normal component is nonzero at the boundary.

An important property of conformal maps is that a function that satisfies Laplace’s equation will continue to do so after a conformal transformation. In other words, if $\nabla^2 V(z)=0$, then provided $w(z)$ is a conformal transformation, $\nabla^2_w V[z(w)]=0$ also. Riemann’s mapping theorem indicates

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the existence of a conformal transformation from a circle to any closed region. So provided a suitable transformation may be found, and provided the boundary conditions map in a simple enough way (as they often do), then it is possible to find solutions in complicated geometries by solving a problem with a simple circular boundary.

III. KARMAN–TREFFTZ TRANSFORMATION

A mapping that may be used to take us from a circle to a shaped cross section with an X-point is the Karman–Trefftz transformation. The Karman–Trefftz transformation is a generalization of the Joukowski transformation that is well known for its use in aerodynamics calculations for the lift from an airplane wing. It maps from a domain that surrounds a circle containing the point $z=-l$ and whose edge passes through $z=l$, to $w(z)$, with

$$
\left( \frac{w + nl}{w - nl} \right)^n = \left( \frac{z + 1}{z - 1} \right)^n.
$$

(1)

For simplicity we will restrict ourselves to domains of $z$ that are symmetric about the real axis, so that $w(z)$ is also symmetric about the real axis with a boundary that has

$$
z = -a + (a + l)\text{e}^{i\alpha}
$$

(2)

with $a > 0$, so that $\alpha$ will parametrize the boundary curve [in both the $z$ and $w(z)$ planes]. Note that $\alpha$ is not the argument of either $z$ or $w(z)$ (if we are centered on $z=a$ in the $z$-plane, $\alpha$ is the poloidal angle). If $n=3/2$, then the cusplike point at $z=l$ becomes an X-point [with a $\pi/2$ interior angle at the joining surfaces in the $w(z)$ plane], and $n=2$ produces the Joukowski transformation. By making $l/a$ arbitrarily small we make the X-point region arbitrarily localized, a situation similar to that described by Webster. This may be seen by rearranging Eq. (1) to give

$$
w(z) = -nl \left( \frac{(z-l)^n + (z+l)^n}{(z-l)^n - (z+l)^n} \right)
$$

(3)

and writing it as an asymptotic expansion in $l/z$, with

$$
w(z) = z + l \left( \frac{1}{z} \right)^{1/2} \left( 1 + \frac{7}{60} \left( \frac{1}{z} \right)^2 + \frac{13}{300} \left( \frac{1}{z} \right)^4 + \cdots \right).
$$

(4)

So provided $|l|/|z|$ is sufficiently small then $w(z)=z$.

In summary, the Karman–Trefftz transformation provides an explicit representation of a transformation from a circular boundary [with $z=-a+(a+l)\text{e}^{i\alpha}$], to a shaped boundary with an X-point.

IV. THE COMPLEX POTENTIAL

We will need to know how the vacuum magnetic field (the gradient of a potential) is transformed as we move from a circular cross section to the X-point geometry. This is most easily accomplished by representing the magnetic field as a complex number whose real and imaginary parts are interpreted as its $x$ and $y$ components, and by defining a complex potential $\Omega$ in terms of the magnetic potential $V$.

The complex representation for the magnetic field is given in terms of the magnetic potential $V$, with

$$
B_z = \frac{\partial V}{\partial x} + i \frac{\partial V}{\partial y}.
$$

(5)

The complex potential $\Omega$ is defined in terms of $V$ and the conjugate function of $V$, such that $\Omega$ is analytic and satisfies the Cauchy–Riemann equations. Specifically,

$$
\Omega(z) = V(z) + i\psi(z)
$$

(6)

with $\psi$ the conjugate function of $V$. Then the Cauchy–Riemann conditions are satisfied with

$$
\frac{\partial V}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial V}{\partial y} = -\frac{\partial \psi}{\partial x}.
$$

(7)

The Cauchy–Riemann conditions may be used to show that

$$
B_w = \frac{d\Omega}{dw},
$$

(8)

and hence the field in the transformed system $w(z)$ may now easily be found from

$$
B_w = \frac{d\Omega}{dz} \frac{dz}{dw} = B_z \frac{dz}{dw}.
$$

(9)

V. THE VACUUM ENERGY

Working in terms of the complex magnetic field and the complex potential, we have

$$
\delta W_V = \int |B_w|^2 dw \cdot dw_w,
$$

(10)

where the integral extends from the boundary that is parametrized by $\alpha$ at $w[z(\alpha)]$ to infinity. To evaluate the integral we use Eq. (9) so that $|B_w|^2 = |d\Omega/dz|^2 |dz/dw|^2$ and we change coordinates back to the circular cross-section coordinates, with $dw/dw_w = [\delta(w_x,w_y)/\delta(x,y)] dx dy$ where $\delta(w_x,w_y)/\delta(x,y) = |dw/dz|^2$ because $w(z)$ is an analytic function. So when we change into the $z$ coordinates [for the purpose of evaluating the integral Eq. (10)], the factors of $|dw/dz|^2$ and $|dz/dw|^2$ cancel to give

$$
\delta W_V = \int |B_z|^2 dx dy.
$$

(11)

In a similar way it may be shown that $\oint B_w \cdot dl_w = \oint B_z \cdot dl_z$, reflecting the fact that the same total current is contained within $l_z$ and $l_w$.

Hence the vacuum energy may be found in terms of the vacuum energy for a solution with a circular boundary, which is much easier to calculate. The actual values of $\delta W_V$ and $B_z$ are determined by the plasma-vacuum boundary conditions that will be modified by the transformation. The mapping of the boundary condition and the resulting boundary condition are obtained in Sec. VI, but for the present we will obtain the general solution in terms of the Fourier coefficients that the boundary conditions will determine.
The vacuum field has \( \nabla \wedge \mathbf{B}_v = \nabla \cdot \mathbf{B}_v = 0 \), so we may write \( \mathbf{B}_v = \nabla V \) with \( \nabla^2 V = 0 \). We will start from a coordinate system with a circular cross section toroidal geometry, then subsequently obtain a two-dimensional (2D) problem by taking the large aspect ratio limit. In this coordinate system,

\[
\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \alpha^2} + \frac{1}{R^2} \frac{\partial^2 V}{\partial \phi^2}
\]

(12)

with \( r, \alpha, \phi \) the radial coordinate, poloidal, and toroidal angle, respectively. Writing

\[
V = \sum_{p=-\infty}^{p=\infty} e^{ipr - \imath n \phi} V_p(r)
\]

(13)

and then projecting out the Fourier components require

\[
0 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_p}{\partial r} \right) - p^2 V_p(r) - n^2 \left( \frac{r}{R} \right)^2 V_p(r)
\]

(14)

which for any given (finite) \( n \) becomes 2D when \( r/R \to 0 \) leaving

\[
0 = r^2 \frac{\partial^2 V_p}{\partial r^2} + \frac{\partial V_p}{\partial r} - p^2 V_p
\]

(15)

that has solutions that tend to zero as \( r \to \infty \), of \( V_p = a_p (r_u/r)^{|p|} \), where the \( a_p \) will be determined by the boundary conditions and \( r_u \) denotes the radial position of the plasma-vacuum surface. Note that it is only because the large aspect ratio limit makes the problem 2D that we are able to use conformal transformations in the calculation.

The vacuum energy is

\[
\delta W_v = \frac{1}{2} \int |\mathbf{B}_v|^2 \, dr
\]

(16)

into which we now substitute \( V \) with

\[
V = \sum_{p=-\infty}^{p=\infty} e^{ipr - \imath n \phi} a_p \left( \frac{r}{r_u} \right)^{|p|}
\]

(17)

and integrate with respect to \( \phi \) and \( \alpha \), with \( r = r_u \) at the surface, to get

\[
\delta W_v = 2 \pi^2 R \sum_{p \neq 0} |p||a_p|^2.
\]

(18)

Here and in the remainder of this article \( R \) will be taken as a typical measure of the major radius that is approximately constant and independent of the poloidal angle. \( R \) is identical to the \( R_0 \) of the first part to this paper, but because the rest of this paper considers a large aspect ratio limit, we will simply write \( R \) as opposed to \( R_0 \).

VI. BOUNDARY CONDITIONS

The boundary conditions in the \( w \)-plane determine \( \tilde{n} \cdot \mathbf{B} \) in terms of the plasma perturbation. We will write \( \tilde{n} \cdot \mathbf{B} \) in the \( w \)-plane as \( n_w \cdot B_w \). However, to obtain the coefficients \( a_p \) we need to know \( \tilde{n} \cdot \mathbf{B} \) in the \( z \)-plane (that we write as \( n_z \cdot B_z \)). Therefore we need to know how \( \tilde{n} \cdot \mathbf{B} \) is transformed as we map between the \( z \)-plane where the boundary is circular with \( z(\alpha) = -a + (a + l) e^{\imath \alpha} \), and the \( w \)-plane whose boundary is shaped and contains an \( X \)-point. This is calculated next.

Recall that the real and imaginary components are considered as orthogonal vector components. Then the tangent vector \( t_w(\alpha) \) of the surface traced by \( w[z(\alpha)] \) is simply given by

\[
t_w(\alpha) = \frac{\partial w[z(\alpha)]}{\partial \alpha}.
\]

(19)

However, using the fact that \( w(z) \) is an analytic function, so that \( \partial w/\partial \alpha = (dw/dz)(\partial z/\partial \alpha) \), then

\[
t_w(\alpha) = \frac{\partial w[z(\alpha)]}{\partial \alpha} = \frac{dw}{dz} \frac{\partial z}{\partial \alpha} = \frac{dw}{dz} t_z,
\]

(20)

where the tangent \( t_z \) of \( z(\alpha) \) in the \( z \)-plane is again simply given by \( t_z(\alpha) = (\partial z/\partial \alpha)/|\partial z/\partial \alpha| \). To obtain the unit normals we rotate the tangent vector by \(-\pi/2\), by simply multiplying by \(-\imath \). Hence,

\[
n_{nw} = -\imath t_w = \frac{\left( w'(z) \right)}{|w'(z)|}(-\imath t_z) = \frac{w'(z)}{|w'(z)|} n_z.
\]

(21)

We already know that \( B_w = B_z d\zeta/dw \), so to obtain how \( n_w \cdot B_w \) transforms we need to simplify

\[
n_w \cdot B_w = \left( \frac{w'(z)}{|w'(z)|} n_z \right) \cdot \left( B_z \frac{d\zeta}{dw} \right),
\]

(22)

where the dot product refers to the sum of the product of the real parts, plus the product of the imaginary parts (examples may be found in Ref. 2). We will use \( d\zeta/dw = 1/(dw/d\zeta) \) \( = (dw/d\zeta)/|dw/d\zeta|^2 \) and write \( n_z = e^{\imath \theta_z} \), \( B_z = rge^{\imath \theta_a} \), and \( dw/d\zeta = r_w e^{\imath \theta_w} \) that substituting into Eq. (22) gives

\[
n_w \cdot B_w = \frac{r_B}{r_w} \left[ e^{\imath \theta_w} e^{\imath \theta_z} \cdot e^{\imath \theta_a} e^{\imath \theta_a} \right] = \frac{r_B}{r_w} \left[ \cos(\theta_w + \theta_z) + \imath \sin(\theta_w + \theta_z) \right]
\]

\[
\cdot \left[ \cos(\theta_w + \theta_B) + \imath \sin(\theta_w + \theta_B) \right]
\]

\[
= \frac{r_B}{r_w} \left[ \cos(\theta_w + \theta_B) \cos(\theta_w + \theta_a) + \sin(\theta_w + \theta_a) \sin(\theta_w + \theta_B) \right]
\]

\[
+ \sin(\theta_w + \theta_a) \sin(\theta_w + \theta_B) \]

\[
= \frac{r_B}{r_w} \cos(\theta_w - \theta_B) = \frac{r_B}{r_w} e^{\imath \theta_w}.
\]

(23)

Greater detail than usual has been given in Eq. (23) due to the unfamiliarity of the “dot product” when applied in the context of complex numbers, and the calculation’s importance in the following work. The last expression in Eq. (23), \( rge^{\imath \theta_a} \), is by the definitions of \( n_z = e^{\imath \theta_z} \) and \( B_z = rge^{\imath \theta_a} \) given above, equal to \( n_z \cdot B_z/|dw/d\zeta| \). Therefore we have
\[ n_w \cdot B_w = \frac{n_z \cdot B_z}{|\frac{\partial z}{\partial r}|}. \] (24)

This calculation is repeated by an alternative method in Sec. IX.

Knowing how \( \mathbf{n} \cdot \mathbf{\dot{B}} \) transforms between \( z \) and the \( w(z) \) plane, we now return to the plasma-vacuum boundary conditions. As shown in Part I, the plasma-vacuum boundary condition is

\[ \nabla \psi \cdot \mathbf{\dot{B}}_{\text{edge}} = \nabla \psi \cdot \mathbf{\dot{B}}^\text{V}_{\text{edge}}, \] (25)

where "edge" refers to the equilibrium position of the surface. Because \( \mathbf{\dot{B}}_0 \cdot \nabla \xi = \nabla \psi \cdot \mathbf{\dot{B}}_1 \), we therefore require that

\[ \mathbf{\dot{B}}_0 \cdot \nabla \xi_{\text{edge}} = \nabla \psi \cdot \mathbf{\dot{B}}^\text{V}_{\text{edge}}. \] (26)

For a single Fourier mode in straight field line coordinates \( \xi_\phi = \xi_\phi(\psi)e^{im\theta} \) with \( \theta = (1/q)\xi v d\chi \), \( q = (1/q)\xi v d\chi \), \( v = \text{I}_2/\text{R}^2 \), and \( \text{J}_2 \) the Jacobian of the orthogonal \( \chi, \psi, \) and \( \phi \) coordinate system, with \( \psi \) the poloidal flux, \( \chi \) the poloidal angle, and \( \phi \) the toroidal angle, and \( I(\psi) = \text{R}_2B_p \), with \( B_\phi \) the toroidal component of the general axisymmetric equilibrium magnetic field \( \mathbf{B} = \nabla \phi + \nabla \phi \times \nabla \psi \). After taking derivatives \( \nabla \psi \cdot \mathbf{\dot{B}}^\text{V}_{\text{edge}} \) then gives

\[ \nabla \psi \cdot \mathbf{\dot{B}}^\text{V}_{\text{edge}} = (im - inq) \frac{I}{q \text{R}^2} \xi_m e^{im\theta - in\phi}. \] (27)

This may alternately be written as

\[ n_w \cdot B_w = im \Delta \frac{\xi_m}{\text{R}_2 \text{B}_p} \frac{I}{q \text{R}^2} e^{im\theta - in\phi}, \] (28)

where \( \Delta = (m - nq)/m \).

Now we transform into the \( z \) coordinates, transforming both \( n_w \cdot B_w = n_z \cdot B_z/|w(z)| \) and \( \text{B}_p = \text{B}_p/|w(z)| \), to get

\[ n_z \cdot B_z = i \Delta \frac{\xi_m}{\text{R}_2} \frac{|w(z)|^2}{\text{B}_p} \frac{I}{q \text{R}^2} e^{im\theta - in\phi}. \] (29)

Using Eq. (13) along with \( V_k = a_k(r_a/|k|) \), and that \( n_z \cdot B_z = \xi - \nabla V \), then gives

\[ \sum_{k = -\infty}^{\infty} a_k |k| e^{\alpha k} = i \Delta \frac{\xi_m}{\text{R}_2} \frac{|w(z)|^2}{\text{B}_p} \frac{I}{q \text{R}^2} e^{im\theta}. \] (30)

From which the Fourier coefficients are easily obtained by multiplying by \( e^{-ip\alpha}/2\pi \) and integrating from \( \alpha = -\pi \) to \( \pi \) to give

\[ a_p = -\left( \frac{\Delta}{|p|} \right) \frac{\xi_m}{\text{R}_2} \frac{1}{2\pi} \int_{\text{B}_p}^{\text{B}_p} \frac{I_r}{|w(z)|^2} e^{im\theta - ip\alpha} d\alpha. \] (31)

In Sec. VII we will see that

\[ \theta(\alpha) = \frac{I_r}{q \text{R}^2} \int_{|w(z)|^2}^{\text{B}_p} \frac{d\alpha}{|w(z)|^2}, \] (32)

which will allow us to integrate by parts once to get

\[ a_p = -\left( \frac{\Delta}{|p|} \right) \frac{\xi_m}{\text{R}_2} \frac{1}{2\pi} \int_{\text{B}_p}^{\text{B}_p} \frac{I_r}{|w(z)|^2} e^{im\theta - ip\alpha} d\alpha. \] (33)

To evaluate the coefficient \( a_p \), we will need an expression for \( \theta(\alpha) \), this is addressed in Secs. VII and VIII.

\[ \theta(\alpha) = \frac{I_r}{q \text{R}^2} \int_{|w(z)|^2}^{\text{B}_p} \frac{d\alpha}{|w(z)|^2}. \] (34)

Hence an element of arc length parallel to the tangent vector, \( dl_w \), has

\[ dl_w = \frac{\partial w}{\partial \alpha} d\alpha = \frac{dw}{dz} \frac{\partial z}{\partial \alpha} d\alpha = \frac{dw}{dz} dl_z. \] (35)

In the absence of equilibrium skin currents, the plasma’s equilibrium field \( \mathbf{\dot{B}}_0 \) equals the vacuum’s equilibrium field \( \mathbf{\dot{B}}^\text{V}_0 \) at the surface between the plasma and the vacuum where the plasma pressure falls to zero, with \( \mathbf{\dot{B}}_{\text{edge}} = \mathbf{\dot{B}}^\text{V}_0 \). Therefore provided that we know the equilibrium vacuum field at the surface, then we also know the plasma’s field at the surface. Consequently, if we know the vacuum field at the surface, then it is possible to calculate the straight field-line variable at the surface. Firstly we note that

\[ \theta = \frac{1}{q} \int^{\chi} \frac{w}{q \text{R}^2} \int^{\chi} \frac{\text{J}_2 + \text{J}_p}{\text{B}_p} = \frac{1}{q \text{R}^2} \int_{\text{B}_p}^{\text{B}_p} dl. \] (34)

Hence an element of arc length \( |dl_w| \) transforms such that

\[ |dl_w| = |dw/\partial z| |dl_z| = |dw/\partial z| r_a d\alpha, \] (34)

as a function of \( \alpha \) in the \( z \)-plane. Using this plus \( |B_w| = |B_z|/|w(z)| \), we may write Eq. (34) as

\[ \theta(\alpha) = \frac{1}{q \text{R}^2} \int_{|w(z)|^2}^{\text{B}_p} \frac{dl_w}{|B_w|} = \frac{I_r}{q \text{R}^2} \int_{|w(z)|^2}^{\text{B}_p} \frac{d\alpha}{|w(z)|^2}. \] (36)

Hence if we know the equilibrium field, then we can obtain an analytical expression for the straight field-line coordinate as a function of \( \alpha \) in the \( z \)-plane.

\[ \theta(\alpha) = \frac{I_r}{q \text{R}^2} \int_{|w(z)|^2}^{\text{B}_p} \frac{d\alpha}{|w(z)|^2}. \] (36)

VIII. EQUILIBRIUM VACUUM FIELD

The equilibrium vacuum field must (i) have a potential that satisfies Laplace’s equation in the vacuum region, (ii) have \( \mathbf{n} \cdot \mathbf{\dot{B}}_0 = 0 \) at the plasma-vacuum boundary (including at the strongly shaped \( X \)-point containing equilibrium), and (iii) have the poloidal field \( B_\phi = 0 \) at the \( X \)-point. The first part is most easily satisfied—we can take a solution that satisfies Laplace’s equation and \( n_z \cdot B_z = 0 \) for a circular cross section, and after a conformal transformation to a shaped cross section we will still have \( n_w \cdot B_w = 0 \) and a potential that satisfies Laplace’s equation. To obtain a field with \( B_\phi = 0 \) at the \( X \)-point, we follow a procedure that is equivalent to that when applying the Kutta condition to obtain the flow around an airplane wing (using a conformal transformation). Essentially, in the \( z \)-plane we combine a homogeneous horizontal field and a circulating field, such that the field becomes zero at a single point on the circular boundary. This is physically equivalent to imposing an external horizontal field, and then driving a current through the plasma. Mathematically it corresponds to taking a complex potential in the \( z \)-plane of...
\[ \Omega = B_{p0} \left\{ i(z + a) - \frac{(a + l)^2}{(z + a)} - 2i(a + l)\ln(z + a) \right\} \]  
\[ \text{with a boundary at} \]
\[ z = -a + (a + l)e^{i\alpha}, \]
\[ \text{with } \alpha \in [0, 2\pi], B_{p0} \text{ a dimensional constant, and the radius of the} \]
\[ \text{circular boundary } r_c = (a + l). \text{The sign of } B_{p0} \text{ determines the direction of} \]
\[ \text{the circulation of } B_{pc}, \text{clockwise} \]
\[ (B_{p0} > 0) \text{ or anticlockwise} \]
\[ (B_{p0} < 0), \text{and the field } B_{pc} \text{ is obtained by} \]
\[ B_{pc} = \frac{d\Omega}{dz} = i(z - l)^2/(z + a)^2 B_{p0}, \]
\[ \text{which at the separatrix given by Eq. (38) gives} \]
\[ |B_{pc}| = 2B_{p0}[1 - \cos(\alpha)]. \text{To consider an outermost flux} \]
\[ \text{surface that is just inside the separatrix, we may instead consider} \]
\[ z = -a + (a + l - \epsilon)e^{i\alpha}, \]
\[ \text{with } \epsilon \ll l. \text{Then for } \epsilon \ll l \ll a \text{ we have} \]
\[ |B_{pc}| \approx 2B_{p0} \left( 1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2} \right) \]
\[ \text{so instead of } B_{pc} = 0 \text{ at the } X\text{-point (that is located at } \epsilon = 0 \text{ and} \]
\[ \alpha = 0), \text{we have } B_{pc} = \epsilon^2/a^2. \text{Notice that we have retained the} \]
\[ \text{singular perturbation in } \epsilon \text{ (singular in that although formally} \]
\[ \epsilon^2/a^2 \ll \epsilon/a, \text{it is the term in } \epsilon^2/a^2 \text{ that qualitatively alters} \]
\[ |B_{pc}| \text{ by preventing it from being zero), further details are given in Ref. 2. The field in the} \]
\[ \text{transformed space is given by} \]
\[ B_{pw} = B_{pc} \frac{dz}{dw}, \text{although we shall not need this here.} \]
\[ \text{Plots of the equilibrium are given in Figure 1.} \]

As we approach the separatrix the behavior of \( \theta(\alpha) \) is dominated by the zeros in \( w'(z) \) and \( B_{pc} \) that occur near the \( X\)-point. Near the \( X\)-point it may be shown (in Ref. 2) that for the case of \( n = 3/2 \) with field lines crossing perpendicularly to each other, \( |w'(z)|^2 \) is given by

\[ |w'(z)|^2 = \frac{1}{\sqrt{2}} \left( \frac{3}{2} \right)^2 \frac{a}{l} \sqrt{1 - \cos(\alpha) + \frac{\epsilon^2}{2a^2}.} \]  
\[ \text{In a similar way to the calculation of Eq. (36), we may rewrite} \]
\[ q = (1/2\pi)\int q = (1/2\pi) \]
\[ \times (I_r/R^2) \frac{\partial |w'(\alpha)|^2}{B_{pc} |q| d\alpha}, \text{then use Eqs. (41) and (42) to find} \]
\[ q = \frac{1}{2\pi R^2 B_{p0}} \int_{\gamma_c} \frac{d\alpha}{\sqrt{1 - \cos(\alpha) + \epsilon^2/2a^2}}. \]  
\[ \text{with } \gamma_c = 2\sqrt{2}(2/3)l(a). \text{Similarly for } \theta(\alpha), \text{using Eqs.} \]
\[ (36), (41), \text{and (42), we get} \]
\[ \theta(\alpha) = \frac{1}{q R^2 B_{p0}} \int_{\gamma_c} \frac{d\alpha}{\sqrt{1 - \cos(\alpha) + \epsilon^2/2a^2}}. \]

Because the integral is dominated by the divergence at the \( X\)-point where \( \alpha = 0, q \) may be approximated by

\[ q \approx \frac{1}{2\pi R^2 B_{p0}} \int_{\gamma_c} \frac{d\alpha}{\sqrt{1 - \cos(\alpha) + \epsilon^2/2a^2}} = \frac{c_\gamma^2}{2\pi} 2 \ln \left( \frac{2a\pi}{\epsilon} \right). \]  
\[ \text{with } c_\gamma = (I_r/R^2 B_{p0})(1/\gamma_c), \text{and similarly} \]
\[ \theta(a) = \frac{c_z}{q} \int_{-\pi}^{\alpha} \frac{d\alpha}{\sqrt{\frac{\alpha^2}{q} + \frac{\alpha}{q} + 1}} = \frac{c_z \sqrt{2}}{q} \ln \left( \frac{\alpha + \sqrt{\alpha^2 + \frac{\alpha^2}{q} + 1}}{-\frac{\alpha}{q} + \sqrt{\frac{\alpha^2}{q} + 1}} \right). \]  

(46)

Therefore as we approach the separatrix, with \( \epsilon \to 0 \), \( q \) has a logarithmic divergence with \( q \sim -\ln(\epsilon) \) that is typical for a tokamak plasma near the separatrix. Hence the qualitative features of \( \theta(a) \) could have reasonably been postulated without showing that they also arise from a vacuum field whose potential satisfies Laplace’s equation and have \( n_w B_{pe}=0 \) on the separatrix, and with \( B_{pe} \to 0 \) at the X-point, but it is reassuring to know that this is also the case.

It is interesting to note that in this model for the equilibrium field, the angle at which the field lines meet at the X-point determines how strongly the divergence is there. For example, if instead of meeting at \( \pi/2 \) the lines make a cusp (tending to parallel as they meet), then \( q \) is finite \([\text{A cusp is obtained by taking } n=2 \text{ in the Karman–Trefftz transformation, Eq. (1)]}\]

IX. MORE TRANSFORMATIONS

Now we consider the calculation of \( \Delta' \), firstly by calculating \( \delta W_V \) from its surface integral representation given in Part I, of

\[ \delta W_V = \pi \int \nabla \psi \cdot \vec{B}_1 \left[ R^2 B_\rho \frac{i}{n} \frac{\partial}{\partial \psi} (\nabla \psi \cdot \vec{B}_1) \right], \]

(47)

where the integral is over the plasma surface. This requires us to know how \( \vec{n} \cdot \nabla \) transforms under a conformal map. The calculation is done partly to reassure us that the expressions we obtain are correct, but also because it is a simple step to subsequently obtain \( \Delta' \).

First we calculate how \( \nabla_z \) transforms. For a real-valued function such as the scalar potential \( V \) defined in Sec. V, for which \( \vec{B}=\nabla V \), we may define the operator \( \nabla_z \) as

\[ \nabla_z V(z) = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) V(z), \]

(48)

where \( z=x+iy \), as an operator analogous to \( \nabla \) of conventional vector analysis. For \( V(z) \) as in Sec. V, \( V \) is a function of Re\( [z(w)] \) and Im\( [z(w)] \), and therefore \( V[z(w)] \) is a function of Re\( [w(z)] \) and Im\( [w(z)] \) as a consequence is also a real function of \( w \). Therefore because \( V[z(w)] \) will be a real-valued function of \( w \), we can also define \( \nabla_w V[z(w)] \) as

\[ \nabla_w V[z(w)] = \left( \frac{\partial}{\partial w_x} + i \frac{\partial}{\partial w_y} \right) V[z(w)], \]

(49)

with \( w=w_x+iw_y \), that again is analogous to the operator \( \nabla \) of conventional vector analysis. Now we will use the chain rule to expand \( \nabla_w V[z(w)] \), noting that because \( w(z) \) is an analytic function it satisfies the Cauchy–Riemann equations, and in addition \( \partial w/\partial x = \partial w/\partial z \). This allows us to show\(^\text{2}\)

\[ \nabla_w V[z(w)] = \frac{\partial w}{\partial z} \nabla_z V[z(w)]. \]

(50)

Note that we could alternately have defined \( \vec{n} \cdot \nabla \), expanded \( \nabla_z V[z(w)] \), then subsequently defined \( \vec{n} \cdot \nabla = (\partial/\partial w_x + i \partial/\partial w_y) \).

Next consider the transformation of \( \vec{n} \cdot \nabla \). For manipulating more complicated expressions, the following identities have been found to be useful:\(^\text{3}\)

\[ a \cdot b c = \bar{a} \cdot \bar{b} c = ab \cdot \bar{c}, \]

(51)

\[ ab \cdot cd = (a \cdot c)(b \cdot d) + (ia \cdot b)(c \cdot d). \]

(52)

For example, to calculate \( n_w B_w \) we use Eq. (52) along with the results of Sec. VI to give

\[ \vec{n} \cdot \vec{B} = n_w B_w = \left( \frac{w'(z)}{|w'(z)|} n_z \right) \cdot \frac{dz}{dw} B_z = \frac{1}{|w'(z)|} \left[ \left( n_z B_z \left( \frac{dz}{dw} + \frac{dz}{dw} \right) \right) + \left( n_z B_z \left( \frac{dz}{dw} - \frac{dz}{dw} \right) \right) \right] = \frac{n \cdot B_z}{|w'(z)|} \]

(53)

as before. In the last step we used,

\[ \frac{dw}{dz} \frac{dz}{dw} = 1, \]

(54)

\[ \frac{dw}{dz} \frac{dz}{dw} = 0 \]

that may easily be confirmed by writing \( dw/dz = \alpha + i\beta \), so that \( \partial z/\partial w = 1/(\alpha + i\beta) \) and multiplying out. A similar calculation for \( \vec{n} \cdot \nabla \) gives

\[ \vec{n} \cdot \nabla = n_w \cdot \nabla_w = \frac{w'(z)}{|w'(z)|} n_z \frac{dz}{dw} \nabla_z = \frac{1}{|w'(z)|} \left[ \left( \frac{dz}{dw} \frac{dz}{dw} \right) (n_z \cdot \nabla_z) \right] + \left( n_z \cdot i \nabla_z \right) \]

(55)

X. RECALCULATING \( \delta W_V \)

Firstly we use the transformed expressions for Eqs. (50), (53), and (55) to rewrite Eq. (47) in coordinates for which the boundary is circular. Then we show that this gives us the same result, Eq. (18) from Sec. V, before showing how the calculation generalizes to give \( \Delta' \) in terms of \( \delta W_V \). These calculations are described in detail in Ref. 2.

Using Eq. (25) and the complex conjugate of Eq. (27), we get
\[ \nabla \psi \cdot \dot{B}_1 |_{\text{edge}} = - \xi_m^e (\psi, \phi) \left( i \frac{I}{qR^2} \right) (im) e^{-im\theta + im\phi}, \]  

(56)

with \( \Delta = (m - nq)/m \). This can be substituted into Eq. (47) to give

\[ \delta W_V = \pi \Delta \left( \frac{m}{nq} \right) \xi_m^i \int \frac{R \int I \dot{\phi}}{B} e^{-im\theta + im\phi} \frac{1}{n} \nabla (RB_n \dot{B}_1), \]

(57)

where we have used \((1/RB_n)\nabla = \partial/\partial \psi\) and \(dl = I_x B_p d\chi\).

Then using \((1/RB_n)\nabla = \partial/\partial \psi\) and \(dl = I_x B_p d\chi\).

Hence to evaluate \( \Delta' \), we need solely to evaluate \( \delta W_V \). \( \Delta' \) may also be evaluated directly, as is also done in Ref. 2, to find

\[ \Delta' = -2 \left\{ \frac{R^2}{|\xi_m^i|^2} \sum_{p \neq 0} |p||a_p|^2 + 1 \right\} \]

\[ = -2 \left\{ \frac{\delta W_V}{2 \pi^2 |\xi_m^i|^2} \right\} \left\{ 1 + O \left( \frac{1}{\delta W_V} \right) \right\}. \]  

(62)

Next we will find that \( \sum_{p \neq 0} |p||a_p|^2 = m|\xi_m^i|^2 \Delta^2/R^2 \) and hence that \( \Delta' = -2m \).

**XI. EVALUATING THE SUM**

Both the vacuum energy \( \delta W_V \) and \( \Delta' \) are determined from \( \sum_{p \neq 0} |p||a_p|^2 \) that we may (in principle) calculate from our analytical expression for \( a_p \). If \( |p| \) is replaced with \( p \) in Eq. (18), then by integrating by parts and noting that \( \theta'(\beta)e^{im\theta(\beta)} = \sum_{p \neq 0} e^{ip\beta}(1/2\pi)\delta \theta'(\alpha)e^{im\theta(\alpha)}e^{-ip\alpha}d\alpha \), we may easily resum the series, finding

\[ \sum_{p = -\infty}^{\infty} |p||a_p|^2 = \Delta^2/|\xi_m^i|^2 R^2. \]  

(63)

The values of the coefficients are expected to be largest for \( p \sim m \sim nq \gg 1 \), and hence we also expect that \( \sum_{p = -\infty}^{\infty} |p||a_p|^2 \approx \sum_{p \neq 0} |p||a_p|^2 = \sum_{p \neq 0} |a_p|^2 \). This has been confirmed by calculating the sums using a saddle point approximation. The result has also been confirmed using a simple model for \( \theta(\alpha) \) that becomes increasingly similar to a step function near a separatrix (as \( q \rightarrow \infty \) the local field line pitch \( \nu \) becoming increasingly peaked near the X-point, so that behaves increasingly like a step function). In this model \( q \sim 1/\delta \) with \( \delta \ll 1 \) and

\[ \theta(\alpha) = \begin{cases} 0 & \alpha \in (-\pi, -\delta) \\ (\alpha + \delta) \frac{2\pi}{\delta} & \alpha \in (-\delta, \delta) \\ 2\pi & \alpha \in (\delta, \pi) \end{cases}. \]  

(64)

Because \( \theta(\alpha) \) is piecewise linear, it is easy to evaluate \( \delta \phi d\alpha e^{im(\theta(\alpha) - ip\alpha)} \) that gives

\[ a_p = -i \frac{p}{|p|} \frac{\xi_m^i}{R} m \frac{\sin(p\delta)}{p(p\delta - m\pi)}, \]  

(65)

for which \( \sum_{p \neq 0} |p||a_p|^2 \) may be evaluated and found to give

\[ \sum_{p = -\infty}^{\infty} |p||a_p|^2 = \Delta^2/|\xi_m^i|^2 R^2. \]  

(66)

It also found that \( \sum_{p = -\infty}^{\infty} |p||a_p|^2 \rightarrow \sum_{p \neq 0} |a_p|^2 \) as \( m \sim nq \rightarrow \infty \). Notice that for \( m \sim nq \gg 1 \) the result of neither approximation methods involve \( \delta \) (or \( e \)), suggesting that the result may be generic and independent of the detailed form of \( \theta(\alpha) \).

Returning to the calculation of \( \Delta' \), Sec. X found that at leading order.
\[ \Delta' = -2 \left( \frac{\delta W_v}{2 \pi^3 \Delta^2} \right). \]  

Therefore using Eq. (62) and \( \Sigma_{p=\infty}^\infty |p| \mu_p^2 = \alpha \Delta^2 \xi_n^2 / R^2 \), we find

\[ \Delta' = -2 \beta. \]  

(68)

A result that appears to be generic for perturbations with \( n \gg 1 \), regardless of whether the plasma cross section is circular or shaped with a separatrix boundary that contains an \( X \)-point. Combining the above result with those from the first part of this paper we find the PM’s growth rate \( \gamma \) has

\[ \frac{\gamma}{\gamma_A} \approx \sqrt{\frac{q}{q^*}} \]  

(69)

as \( q^* \rightarrow \infty \) near a separatrix, with \( \gamma_A = B^2 / (\rho_0 R \delta d) \). Therefore as the plasma-vacuum boundary more closely approximates a separatrix because \( q^* \rightarrow \infty \) more rapidly than \( q \), we find the growth rate has \( \gamma / \gamma_A \rightarrow 0 \) indicating that our model for the linear stability of the high-\( n \), ideal magnetohydrodynamic (MHD) PM, predicts it to be marginally stable for plasmas whose boundary has a separatrix with at least one \( X \)-point.

**XII. DISCUSSION—PREVIOUS WORK**

As mentioned at the outset, Laval et al.\(^5\) considered a trial function consisting of a single Fourier mode in a straight field line coordinate that is resonant at a rational surface in the vacuum just outside the plasma’s surface. For that trial function, they found that for a positive nonzero current at the plasma edge, \( \delta W < 0 \), and suggested therefore that the PM would be unstable for a nonzero positive current at the plasma’s edge. On the basis of the sign of \( \delta W \) our study also finds this, but our study also suggests that the growth rate will asymptote to zero as the outermost flux surface approximates a separatrix, so that the mode will be marginally stable.

Lortz\(^6\) also considered PM stability in toroidal plasmas with shaped cross sections, using a systematic calculation with a trial function whose resonant surface is inside the plasma. An advantage of the calculation by Lortz\(^6\) is that the radial structure of the mode is considered. An unfortunate complication for this discussion is that in the ordering scheme of Lortz,\(^6\) the vacuum energy can be neglected. This was not the case for our calculation in Part I\(^7\) or of Laval et al.\(^5\) Nonetheless, we will consider the predictions of this calculation in the limit where the outermost flux surface approximates a separatrix.

Connor et al.\(^8\) review the calculation of Lortz\(^6\) and use it to consider trial functions with resonant surfaces both inside and outside the plasma. They find that stability of the PM requires

\[ 1 - 4D_M > \left( \frac{2}{P} - 1 \right)^2, \]  

(70)

where the Mercier coefficient \( D_M = -Q / P \), and \( P \), \( Q \), and \( S \) are defined as

\[ P = 2\pi(q')^2 \left[ \int \frac{J_x B^2}{R^2 B_p^2} d\chi \right]^{-1}, \]

\[ Q = \frac{p'}{2\pi} \int \frac{\partial J_x}{\partial \phi} d\chi - \frac{(p')^2}{2\pi} \int \frac{J_x}{B_p} d\chi + Ip' \left[ \frac{1}{2\pi} \int \frac{J_x B^2}{R^2 B_p^2} d\chi \right]^{-1} \times \left[ \frac{1}{2\pi} \int \frac{J_x B^2}{R^2 B_p^2} - q' \right], \]  

(71)

\[ S = P + q' \left[ \frac{1}{2\pi} \int \frac{J_x B^2}{R^2 B_p^2} d\chi \right]^{-1}. \]

Substituting \( D_M = -Q / P \) into Eq. (70) gives the stability requirement

\[ \frac{Q}{P} - \left( \frac{S}{P} \right)^2 + \frac{S}{P} > 0. \]  

(72)

Substituting for \( P \), \( Q \), and \( S \) allows Eq. (72) to be simplified to

\[ p' \left[ \frac{1}{2\pi} \int \frac{\partial J_x}{\partial \phi} d\chi - p' \frac{1}{2\pi} \int \frac{J_x}{B_p} d\chi \right] + q' \left[ \frac{1}{2\pi} \int \frac{J_x B^2}{R^2 B_p^2} d\chi - q' \frac{1}{2\pi} \int \frac{J_x B^2}{R^2 B_p^2} d\chi \right] > 0. \]  

(73)

This may be simplified further by noting that because \( \nu = J_x / R^2 \), and in a large aspect ratio ordering where \( R \) is taken as approximately constant,

\[ \frac{\partial}{\partial \psi} \frac{1}{2\pi} \int \frac{J_x d\chi}{R^2} = \frac{\partial}{\partial \psi} \frac{1}{2\pi} \int \frac{\nu d\chi}{I} = \frac{\partial}{\partial \psi} \frac{1}{2\pi} \int \frac{\nu d\chi}{R^2} - \frac{R^2}{I} \frac{1}{2\pi} \int \frac{\partial R}{\partial \psi} d\chi \]  

(74)

and

\[ q' = \frac{1}{2\pi} \int \frac{\partial \nu}{\partial \psi} d\chi. \]  

(75)

The Grad–Shafranov equation in \( \psi \), \( \chi \), and \( \phi \) coordinates has

\[ \frac{\partial \nu}{\partial \phi} = \frac{\nu}{B_p^2} - \frac{\partial \nu}{\partial \psi} \left( p + B^2 \right) + \frac{R^2 B^2}{I} \frac{\partial}{\partial \psi} \left( \frac{1}{I} \right). \]  

(76)

Therefore if \( R \) is taken as approximately constant (as would be the case either in a large aspect ratio limit or if we are sufficiently close to the separatrix that the integral is dominated by the divergence at the \( X \)-point and \( R \) may be approximated by its value \( R_X \) there), then using Eq. (76), Eq. (73) simplifies to a condition for stability of
0 < \frac{1}{2\pi} \int \frac{\nu}{B_p^2 R^2} d\chi \left\{ -\rho' - \frac{2H}{R^2} \frac{\partial}{\partial \psi} (2\rho + B^2) \right\} \\
= (\nabla \phi \cdot \vec{J}) \frac{1}{2\pi} \int \frac{\nu}{R^2 B_p^2} \frac{\partial}{\partial \psi} (2\rho + B^2) d\chi. \quad (77)

This may be simplified further still to

0 < \frac{1}{2\pi} \int \frac{\nu}{B_p^2 R^2} \left( -\nabla \phi \cdot \vec{J} \right) \left[ 2(\nabla \phi \cdot \vec{J}) - \frac{\partial B_p^2}{\partial \psi} \right] d\chi. \quad (78)

Because the integrals are dominated by the divergence of \( \nu/B_p^2 \) at the X-point and \( \partial B_p^2/\partial \psi \) at the X-point, then based on the formulation of Lortz,\(^6,8\) then provided \( \nabla \phi \cdot \vec{J} > 0 \) the negative expression clearly indicates instability to the PM. However, if we allow \( \nabla \phi \cdot \vec{J} \) to be negative, then the formulation of Lortz\(^6,8\) also suggests that stability is possible provided

0 < \frac{1}{2\pi} \int \frac{\nu}{B_p^2 R^2} \left[ -2(\nabla \phi \cdot \vec{J}) - \frac{\partial B_p^2}{\partial \psi} \right] d\chi. \quad (79)

Therefore in principle there is a range of negative current values at the plasma edge for which the PM is stable. In Ref. 2 we have calculated \( \partial B_p^2 / \partial \psi \) near the X-point for a standard and a “snowflake”\(^9\) divertor, finding that near the separatrix \( \partial B_p^2 / \partial \psi \) tends to a constant for a conventional divertor but tends to zero for a snowflake divertor. An interesting consequence of this is that a conventional X-point has a range of negative current values for which the PM is stable, but in the limit of an exact snowflake X-point (with flux surfaces meeting at an angle of \( \pi/3 \)), the range of values of negative current for which the PM is stable tends to zero. Whether this observation will have consequences for the plasma behavior in a snowflake X-point geometry remains to be seen, but it is a qualitative difference between a conventional X-point and that produced with a snowflake divertor.

As mentioned previously, the calculation also considered the radial structure of the mode, with \( \xi \sim x^\lambda \), \( x \) a radial coordinate, and

\[
\lambda = -1/2 \pm \sqrt{\frac{1 + Q}{4} + \frac{Q}{P}}. \quad (80)
\]

As we approach the separatrix, Webster\(^4\) shows that \( D_M = -\frac{Q}{P} \to 0 \), giving

\[
\lambda_+ = \frac{Q}{P} \to 0, \quad (81)
\]

and mode structures of \( \xi_+ \sim 1/x \) and \( \xi_- \sim x^{Q/P} \). For the perturbations to satisfy the boundary condition of a mode amplitude that tends to zero in the plasma, this requires us to use \( \xi_- \) for resonances outside the plasma (the “external” PM) and \( \xi_+ \) for resonances inside the plasma (the “internal” PM).

It should be noted that a potential problem with the analysis of Lortz\(^5\) when applied to PMs, that the mode is taken to be sufficiently localized that the equilibrium quantities (that include \( q \) and \( q' \)), is approximately constant. This is almost certainly not the case near a separatrix.

### XIII. Conclusions

The first part of this paper generalized a simple model for the PM to toroidal tokamak geometry. It found that PM stability is determined by the value of \( \Delta' \), a normalized measure of the discontinuity in the gradient of the normal component of the perturbed magnetic field at the plasma-vacuum boundary. This paper calculated \( \Delta' \) for a large aspect ratio Tokamak geometry with a separatrix and X-point, in such a way that the effect of the X-point is captured exactly, thereby avoiding the numerical divergences that can occur near an X-point. This was possible by generalizing the method of conformal transformations beyond textbook presentations that require a boundary condition of either the function or its normal derivative to be zero. Here it is shown that a conformal transformation can be used even if a field’s normal derivative is nonzero and continuously varying at the boundary. In this case instead of obtaining an exact analytic solution (as would be the case if its normal derivative were zero on the boundary), the 2D problem is reduced to a one-dimensional problem that may subsequently be solved exactly or approximately. The approach avoids the errors that may arise due to the discretization of space near an X-point that are necessarily present in most numerical methods. This paper also calculated analytical expressions for physically realistic examples of the equilibrium vacuum magnetic field, and the straight field line angle at the plasma-vacuum boundary. These and other results are likely to find opportunities for application elsewhere. For a plasma perturbation consisting of a single Fourier mode in straight field line coordinates with a high toroidal mode number \( n \) (the trial function found unstable by Laval et al.\(^5\)), in a plasma equilibrium with a separatrix and an X-point in a large aspect ratio approximation, it is found that the change in the vacuum energy is identical to that for an equivalent perturbation in a cylindrical equilibrium, with \( \delta W_V = 2\pi^2 (|\xi_m|^2/R) \Delta^2 m \). The resulting value of \( \Delta' \) is also found to be the same, with \( \Delta' = -2m \) where \( m \) is the poloidal mode number.

Previous works by Lortz\(^5\) and Connor et al.\(^8\) were considered in the context of a plasma boundary that tends to a separatrix with an X-point. Like Laval et al.\(^5\), their work predicts the PM to be unstable if there is a positive current at the plasma’s edge, and it also finds a well behaved radial structure for the mode. Interestingly, for a conventional X-point there is predicted to be a range of small but negative edge-current for which the PM is stable, but in the limit of an exact snowflake divertor this range shrinks to zero size—a qualitative difference between a conventional and a snowflake divertor. A limitation of the Lortz calculation is that it approximates the equilibrium quantities as constant on the length scale of the plasma instability; this is not necessarily the case for \( q \) or \( q' \) near a separatrix, and therefore the results when applied to a separatrix case should be treated with caution. Likewise, as noted in Part I, there are potential limita-
tions to the high-$n$ ordering form of $\delta W$ used here, and this should be investigated in future work.

In conclusion, we have developed a simple model for the PM and found that despite $\delta W < 0$, the growth rate $\gamma$ tends to zero as the outermost flux surface more closely approximates a separatrix with an $X$-point. As the outermost flux surface approaches a separatrix, the growth rate falls with $\ln(q'/q); \; \text{this has subsequently been confirmed with ELITE,}^{10,11}$ leading us to believe that the effect of a separatrix on the high toroidal mode number ideal MHD model is now understood. The ideal MHD prediction of marginal stability at the separatrix means that other nonideal terms such as resistivity, nonlinear terms, or terms neglected in the high-$n$ analysis, will play a role in determining the eventual stability. In general it is hoped that the methods and results contained in this paper will provide new tools for studying plasmas in separatrix geometries and have potential applications in future studies of plasma stability and more generally outside of plasma physics.

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2. See EPAPS supplementary material at http://dx.doi.org/10.1063/1.3194271 for full detailed descriptions of the calculations outlined in this article.