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Asymmetric tearing mode in the presence of viscosity

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The linear stability of the tearing mode (TM) in a plasma column is investigated in the presence of viscosity and finite equilibrium current density gradients (i.e., asymmetries). It is shown that for low β, both effects are essential in order to properly describe the mode behaviour close to marginality. In particular, the theory introduces a critical threshold for the destabilization, such that the perturbation grows only if $\Delta' > \Delta'_{cr}$. The value of Δ'_{cr} depends on the equilibrium configuration and on the plasma parameters. Most importantly, Δ'_{cr} can take negative values, thus allowing unstable tearing modes for $\Delta' < 0$ (even in the absence of bootstrap current). [doi:10.1063/1.3660410]

I. INTRODUCTION

Magnetic reconnection is a pervasive phenomenon in both natural and experimental plasmas[1] through which large amounts of magnetic energy are converted and released in the form of kinetic and thermal energy. In magnetically confined plasmas, it can lead to the appearance of structures that can reach a macroscopic scale, the magnetic islands, and limit the performance of the experiment or even cause disruptive events, with a sudden loss of the confinement.

In the comprehensive work by Furth *et al*.,[2] the instability at the origin of the magnetic island was identified as the tearing mode (TM). In its classic formulation, this is a resistive linear perturbation driven unstable by current (and also pressure) gradients. The smallness of the island with respect to the equilibrium scale allows a perturbative treatment of the problem, and the stability of the mode is determined through a matching technique. The external MHD region is connected to the thin resistive layer through a stability parameter, Δ', a positive value of which implies instability (a second parameter, A, is introduced in asymmetric cases[3]). Unfortunately, the evaluation of this parameter is everything but straightforward, as proved in several works.[4–6]

However, it was soon recognized[7] that, despite its usefulness in understanding the instability threshold, the linear theory breaks down for extremely small island widths (compared to the scale of the confined plasma) and is, therefore, inadequate to properly describe the dynamics of the TM in magnetic fusion devices. To overcome this problem, an elegant and fully nonlinear treatment, based on the decoupling the magnetic and velocity fields and on the smallness of Δ', was introduced by Rutherford.[7] The scalar relation derived in Ref. 7 and representing the evolution of the magnetic island width in its nonlinear phase is commonly known as Rutherford equation (RE).

Throughout the years, several relevant physical effects were included in the model in the form of extra terms in the so-called generalized Rutherford equation (GRE). A particularly important addition was the introduction of the effect of the Bootstrap current,[8,9] which provides a destabilization mechanism directly driven by the pressure gradients. This has strong consequences for magnetically confined plasmas, since to avoid the presence of magnetic islands, the tailoring of the current density profile is not sufficient and it has to be combined with pressure control. This, in turn, implies that β (the ratio between kinetic and magnetic pressure), which is a measure of the plasma performance, has to be limited. This, of course, is a cause of major concern for the next generation tokamaks, such as ITER (Ref. 10) and DEMO, which will need active control techniques. However, this new instability, renamed neoclassical tearing mode (NTM), has a metastable nature as it is linearly stable and is excited only in the presence of a finite size seed island, which could be caused by turbulence, or other instabilities such as sawteeth, ELMs (Edge Localized Modes), or fishbones.

Recent experimental observations in DIII-D (Ref. 11) operating in ITER like conditions cast some doubts on the general validity of the standard NTM model. Indeed, the appearance of an island was reported to depend on the current density rather than on the pressure profile, implying that the magnetic instability could be triggered by a classical mechanism (i.e., the bootstrap current contribution might not be essential). These results suggest that a better understanding of the NTM as well as the TM theory is required. Indeed, several aspects of the problem are not fully clarified and some basic questions remain open.

The aim of this work is to shed some light on the specific issue of the combined effect of the asymmetry and the viscosity on the linear phase of the TM. In particular, the asymmetry that we are describing arises from the finite gradients of the equilibrium current density at the resonant surface, which introduce new terms in the model equation and break the parity of the solutions. These terms naturally occur in toroidal and in cylindrical geometry, where the equilibrium current density has typically finite gradients everywhere but on the magnetic axis (excluding advanced scenarios). In this paper, we focus on the cylindrical configuration, i.e., we assume a straight tokamak approximation in which mode coupling is avoided. Several inviscid theoretical models have established that these geometrical effects modify both the linear growth rate of the mode (by changing the dispersion relation in low β cases)[3] and its nonlinear evolution (by adding extra terms in the GRE).[12–14] Here, we study how these results are affected by a small but finite viscosity.

II. MODEL AND EQUATIONS

In our investigation, we use the reduced resistive and viscous magneto-hydrodynamic model.[15] Its equations are appropriate to describe low $\beta = 4\pi nT/B^2$ plasmas with small pressure gradients and evolve the normalized electric potential, ϕ, which also represents the stream function of the plasma and the normalized magnetic flux, ψ. These scalar fields are related to the normalized magnetic and velocity vectorial fields through: $\mathbf{B} = \mathbf{e}_z + \nabla\psi \times \mathbf{e}_z$ and $\mathbf{V} = \nabla\phi \times \mathbf{e}_z$. Here, $\mathbf{e}_z$ is the unity vector along the ignorable coordinate (i.e., the axis of the cylinder in cylindrical geometry). The magnetic field, the velocity, and the lengths are normalized with respect to the value in the z direction, B_0, the Alfven velocity, $v_A = B_0/\sqrt{4\pi n m_i}$, and an equilibrium length scale L, respectively. As a consequence, the magnetic flux is normalized with respect to B_0/L and the electric potential with respect to $v_A B_0 L/c$ (c is the speed of light). Finally, we assume that the equilibrium is axisymmetric and that the parallel velocity is always negligible (i.e., the velocity is given only by the $\mathbf{E}\times\mathbf{B}$ drift). The model equations, representing quasi neutrality ($\nabla\cdot\mathbf{J} = 0$) and the parallel component of Ohm's law are

$$\frac{\partial \nabla_\perp^2 \phi}{\partial t} + \mathbf{V}\cdot\nabla(\nabla_\perp^2\phi) = \mathbf{B}\cdot\nabla J + \mu\nabla_\perp^4\phi, \tag{1}$$

$$\frac{\partial\psi}{\partial t} + \mathbf{B}\cdot\nabla\phi = -\eta(J - J_{eq}). \tag{2}$$

Here, η is the normalized resistivity (i.e., the inverse magnetic Reynolds number) and μ is the normalized viscosity such that $\mu = P\eta$ where P is the magnetic Prandtl number. The system is closed by Ampere's law, which relates the normalized parallel current density, J, to the normalized magnetic flux

$$J = -\nabla^2\psi. \tag{3}$$

In cylindrical geometry, the above operators become: $\mathbf{V}\cdot\nabla = r^{-1}(\partial_r\phi\partial_\theta - \partial_\theta\phi\partial_r)$, $\mathbf{B}\cdot\nabla = -r^{-1}(\partial_r\psi\partial_\theta - \partial_\theta\psi\partial_r) + \partial_z$, and $\nabla^2 = r^{-1}\partial_r(r\partial_r) + r^{-2}\partial_\theta^2$, where r and θ are the radial and angular coordinates.

The equations can be linearised around an axisymmetric equilibrium without electric drift flows: $\psi_{eq} = \psi_{eq}(r)$, $\phi_{eq} = 0$ with a perturbation in harmonic form $\tilde\psi(r,\theta,z,t) = \sum_{m,n}\Psi_{m,n}(r)e^{\gamma t + i[m\theta - (n/R)z]}$. Here, γ, m, and n are the growth rate and the two mode numbers while, assuming that L is the minor radius and R is the aspect ratio of the cylindrical tokamak (remember that we are working with dimensionless variables). In the following, we drop the subscripts m and n for the sake of simplicity. It is well known that the resistivity is important only in a thin layer around the resonant surface, r_s, which is formed where $F(r_s) = 0$, with $F(r) \equiv (m/r)d\psi_{eq}/dr + n/R$. Using the Fourier representation of the perturbations, the previous equations become

$$\frac{\gamma r_s^2}{\eta}\left[\Psi\left(1 + i\frac{\omega_*}{\gamma}\right) + \hat F(\chi)(\Xi + \frac{ik}{\gamma}N)\right] = \Psi'' + \frac{\Psi'}{1+\chi} - \frac{m^2}{(1+\chi)^2}\Psi, \tag{4}$$

$$\frac{\gamma^2}{k^2}\left[\Xi'' + \frac{\Xi'}{1+\chi} - \frac{m^2}{(1+\chi)^2}\Xi\right] = \hat F(\chi)\left\{\Psi'' + \frac{\Psi'}{1+\chi} - \left[\frac{m^2}{(1+\chi)^2} - \frac{m}{1+\chi}\frac{J'_{eq}(\chi)}{k\hat F(\chi)}\right]\Psi\right\} + \frac{\mu\gamma}{k^2 r_s^2}\nabla^4\Xi, \tag{5}$$

where we have defined $\chi \equiv (r - r_s)/r_s$, $\Xi \equiv -ik\Phi/\gamma$ (the plasma displacement), $k \equiv ns/R$ [$s \equiv r(dq/dr)/q$ is the magnetic shear and $q \equiv -r/(R d\psi_{eq}/dr)$ is the safety factor], $\hat F(\chi) = F(r_s + r_s\chi)/k$, and the prime represents a derivative with respect to χ. Finally, we define the operator $\nabla^4\Xi \equiv \Xi^{IV} + 2\Xi'''/(1+\chi) - (1+2m^2)\Xi''/(1+\chi)^2 + (1+2m^2)\Xi'/(1+\chi)^3 + (m^2 - 4m^4)\Xi/(1+\chi)^4$.

The problem can be made more tractable by solving simplified equations valid in different (but overlapping) regions of the plasma. In particular, we define the "outer" region as the part of the integration domain that is sufficiently far from the resonant surface so that the plasma can be considered ideal. Outside the non-ideal layer, but in proximity of the resonant surface, the equation for the outer solution can be obtained by expanding Eq. (5)

$$\Psi'' + (1-\chi)\Psi' - \left(\hat k^2 - \frac{a}{2} + \frac{3}{2}a^2 + \frac{a}{\chi}\right)\Psi = 0, \tag{6}$$

with the coefficients $\hat k^2 \equiv m^2 + b + a - 2a^2$, $a \equiv (J'_{eq}/J_{eq})(1 - 2/s)$, and $b \equiv (J''_{eq}/J_{eq})(1 - 2/s)$ (evaluated at the rational surface) determined only by the equilibrium configuration of the parallel current density. Similarly, Eq. (4) reads

$$\chi\Xi = -\Psi + \left(\frac{\gamma}{\eta r_s^2}\right)^{-1}[\Psi'' + (1-\chi)\Psi' - m^2\Psi], \tag{7}$$

where we allow for vanishing growth rates, which implies that the dominant balance could be given also by the second term on the right-hand side (note that Ξ scales inversely with γ). Note also that $\hat F(\chi) \cong \chi$ for $\chi \to 0$ was used on the left-hand side.

Equation (6) is now a single second order ordinary differential equation (ODE), the solution of which can be obtained in terms of Froebenius series (the equation is singular at $\chi=0$): $\Psi=\sum_{k=0}^{\infty}[\alpha_k\chi^k+\beta_k\chi^k\log(|\chi|)]$. Substituting this expression in Eq. (6) gives

$$\Psi^{\pm}=\left\{1+a\chi\log(|\chi|)+(a^2-a)\frac{\chi^2}{2}\log(|\chi|)+\frac{\hat{k}^2\chi^2}{2}+\cdots+\frac{A\pm\Delta'}{2}\left[\chi+\frac{(a-1)\chi^2}{2}+\cdots\right]\right\}\Psi_0. \tag{8}$$

Due to the singularity, the solution consists of two independent branches to the left and to the right of the resonant surface, which we have indicated as Ψ^+ and Ψ^-. A solution can be obtained assuming that both branches reach the same value Ψ_0 from the left and from the right of the singularity. The remaining two boundary conditions are set once the parameters Δ' and A are assigned (in order to do that, a full numerical solution of the equation $\mathbf{B}\cdot\nabla J=0$ is necessary). To avoid confusion, it is useful to remark that our Δ' is normalized, so that $\Delta'=r_s d(\log\Psi)/dr|_{r_s^-}^{r_s^+}$. A more detailed derivation of the outer solution in the presence of asymmetric terms (i.e., a and A different from zero) can be found in Ref. 13. Once the magnetic flux in the outer region is obtained, it is trivial to calculate the plasma displacement using Eq. (7).

We move now to the inner region, where the non-ideal effects dominate the plasma dynamic. The thin layer around the resonant surface has a width of order δ, which is typically small as it is proportional to some power of the resistivity, depending on the regime considered. The inner region is, therefore, better described once the perpendicular coordinate is rescaled with respect to δ, $z\equiv\chi/\delta$. The relevant equations thus become

$$\frac{\gamma\delta^2 r_s^2}{\eta}[\hat{\psi}+z\xi]=\hat{\psi}''+\delta(1-\delta z)\hat{\psi}'-\delta^2 m^2\hat{\psi}, \tag{9}$$

$$\frac{\gamma\delta^2 r_s^2}{\eta}\xi''=\frac{\gamma\delta^2 r_s^2}{\eta}\frac{\delta^2 k^2}{\gamma^2}z\left\{\hat{\psi}''+\delta(1-\delta z)\hat{\psi}'-\delta^2\times\left[m^2+b-\frac{a^2}{2}+\frac{a}{2}+\frac{a}{\delta z}\right]\hat{\psi}\right\}+\frac{\mu}{\eta}\xi^{IV}, \tag{10}$$

where $\xi=\delta\Xi/\Psi_0$, $\hat{\psi}=\Psi/\Psi_0$, and the prime now represents derivatives with respect to z.

The global solution is determined once the inner and the outer solutions are matched. In particular, the dispersion relation is obtained through the condition

$$\delta\Delta'=\frac{1}{\Psi_0}\int_{-\infty}^{\infty}\frac{\partial^2\Psi}{\partial z^2}dz, \tag{11}$$

where the right-hand side is calculated with the fundamental harmonic of the inner magnetic flux perturbation.

III. LAYER SOLUTION

In this section, we investigate how the viscosity affects the linear dispersion relation in the presence of an asymmetric equilibrium. Grasso *et al.*[16] showed that, in symmetric equilibria, the viscous term generates a critical threshold for the excitation of the mode, i.e., Δ' has to exceed a certain value (typically, but not necessarily greater than zero) in order for the tearing mode to become unstable. Similarly, Militello *et al.*[3] have shown[3] with an inviscid calculation that close to marginality the equilibrium asymmetries significantly change the dispersion relation of the mode. In the following, we combine the two theories in order to produce a visco-asymmetric theory of the linear tearing mode.

Our starting point are Eqs. (9) and (10), from which we can obtain an expression for the plasma displacement as a function of the magnetic flux only (i.e., without its derivative)

$$\hat{P}\xi^{IV}-\hat{Q}\xi''+z^2\xi=\left[\hat{A}-z\left(1-\hat{B}\right)\right]\hat{\psi}. \tag{12}$$

Here, $\hat{P}=P/R$, $\hat{Q}=Q/R$, $\hat{A}=a\delta/Q$, $\hat{B}=\delta^2(b-a^2/2+a/2)/Q$, and $P=\mu/\eta$ is the Prandtl number, $Q=\gamma\delta^2 r_s^2/\eta$ and $R=Q^2\delta^2k^2/\gamma^2$. Equation (12) can be simplified if appropriate orderings are identified.

In the classic treatment,[2,17] the equilibrium terms $\hat{A}$ and $\hat{B}$ are neglected and we obtain the resistive tearing mode when $P\ll Q$ and the visco-resistive tearing mode in the opposite limit. In the former case, the dominant balance is between the second and the third term on the right-hand side of Eq. (12), so that $\delta\sim(\gamma\eta)^{1/4}/(r_s k)^{1/2}\equiv\delta_\eta$, while in the latter, the balance is between the first and third term, which gives $\delta\sim(\mu\eta)^{1/6}/(r_s^2 k)^{1/3}\equiv\delta_\mu$. In both cases, we limit our attention to slowly growing modes for which $Q\ll 1$ (i.e., perturbations relatively close to marginality), so that the (resistive or ideal) kink instabilities are excluded from our treatment. Using the matching condition, Eq. (11), together with Eq. (9) and the constant-ψ approximation,[2] it is easy to obtain that $\gamma\sim\Delta'\eta r_s^{-2}\delta^{-1}$. Hence, for the resistive and visco-resistive regimes, we retrieve the standard dispersion relations: $\gamma\sim\eta^{3/5}\Delta'^{4/5}k^{2/5}r_s^{-6/5}\equiv\gamma_\eta$ and $\gamma\sim\eta^{2/3}P^{-1/6}\Delta' r_s^{-4/3}k^{1/3}\equiv\gamma_\mu$, respectively. Replacing the former expression in the resistive layer width gives: $\delta_\eta\sim\eta^{2/5}\Delta'^{1/5}r_s^{-4/5}k^{-2/5}$ (δ_μ does not depend on γ).

We can now evaluate when the terms proportional to $\hat{A}$ and $\hat{B}$ become important. In order to do that, we approximate $\hat{\psi}$ using Eq. (8), a technique already used in Ref. 3 and that will be rigorously justified in the following. We thus find that $\hat{A}\hat{\psi}\sim\delta\log\delta/\Delta'$, while $\hat{B}\hat{\psi}\sim\delta/\Delta'$, implying that $\hat{A}$ generally dominates over $\hat{B}$. From this observation, we can expect that the asymmetric terms will significantly affect the linear threshold, which will differ from that predicted in Ref. 16, where only the b contribution of the term $\hat{B}$ was considered. We, therefore, find that the asymmetric equilibrium terms have to be taken into account when $\Delta'\sim\delta\log\delta$ for both the resistive and visco-resistive regime.

A proper treatment of the tearing mode close to marginality must, therefore, include the effect of the asymmetries. At the same time, it is easily verified that for small growth rates, the plasma is in the visco-resistive regime. Indeed, when $P\gtrsim Q$, which can be rewritten as $\gamma\lesssim P^{2/3}\eta^{1/3}k^{2/3}r_s^{-2/3}$, the viscous effects become important. These observations lead to the conclusion that, within our model, a realistic prediction of the linear threshold can be obtained only in the framework of the visco-asymmetric regime.

In order to obtain a dispersion relation in this limit, we employ an extended version of the constant-ψ approximation introduced in Ref. 3. We, therefore, assume that the structure of the inner solution resembles that of the outer solution (see Eq. (8)), so that

$$\hat{\psi} = 1 + \delta_\mu z\{c_1 + c_3[\log(1+z^2) + 2\log\delta_\mu]\} + \delta_\mu^2 z^2\{c_2 + c_4[\log(1+z^2) + 2\log\delta_\mu]\} + \hat{\psi}_1(z), \quad (13)$$

where $c_1 \equiv A/2$, $c_2 \equiv \hat{k}^2/2 + A(a-1)/4$, $c_3 \equiv a/2$, $c_4 \equiv (a^2 - a)/4$, and $\hat{\psi}(z)$ is a small correction of order $O(\delta_\mu \Delta')$ which smoothly matches to the outer solution. Replacing Eq. (13) in Eq. (12) and keeping only the lowest order contributions in δ_μ, we find

$$\xi^{IV} + z^2\xi \cong -z + \frac{a}{\Delta} + z\mathcal{F}(z)\frac{\delta_\mu}{\Delta}, \quad (14)$$

where

$$\mathcal{F}(z) \equiv b - \frac{a^2}{2} + \frac{a}{2} + \frac{aA}{2} + a^2\log\delta_\mu + \frac{a^2}{2}\log(1+z^2), \quad (15)$$

we have used the fact that $\hat{P} = 1$ when $\delta = \delta_\mu$, and we have defined $\Delta \equiv \gamma\delta_\mu r_s^2/\eta$ as it scales like Δ' when the equilibrium terms are not included in the calculation.

The solution of Eq. (14) can be written as: $\xi = \xi_1 + \xi_2 + \xi_3$, where

$$\hat{\xi}_1^{IV} + z^2\hat{\xi}_1 = -z, \quad (16)$$

$$\hat{\xi}_2^{IV} + z^2\hat{\xi}_2 = z\log(1+z^2), \quad (17)$$

$$\hat{\xi}_3^{IV} + z^2\hat{\xi}_3 = 1, \quad (18)$$

where $\hat{\xi}_1 \equiv \xi_1/[-1 + (\delta_\mu/\Delta)(b - a^2/2 + a/2 + a^2\log\delta_\mu)]$, $\hat{\xi}_2 \equiv \xi_2/[(\delta_\mu/\Delta)a^2/2]$, and $\hat{\xi}_3 \equiv \xi_3/(a/\Delta)$. Note that ξ_1 and ξ_2 are odd functions of z, while ξ_3 is even. This observation is important since only the odd part of the displacement plays a role in the matching with the outer solution and, therefore, in the definition of the dispersion relation. Indeed, using Eq. (13), the matching is automatically obtained for most of the terms and Eq. (11) reduces to: $\delta\Delta' = \int_{-\infty}^{\infty}\hat{\psi}_1'' dz$. This expression can be rewritten in a more convenient form by employing Eq. (9)

$$\Delta\int_{-\infty}^{\infty}(1 + z\xi)dz - \delta_\mu\int_{-\infty}^{\infty}\mathcal{F}(z)dz + \delta_\mu\pi a^2 = \Delta'. \quad (19)$$

If we now split ξ, we find that its even part, ξ_3, does not contribute to the integral, while the remaining terms give the dispersion relation

$$\alpha_1\frac{\gamma r_s^2}{\eta} = \frac{\Delta'}{\delta_\mu} + \alpha_1\left[\frac{Aa}{2} + a^2\log\delta_\mu - a^2\frac{\alpha_2+\pi}{\alpha_1} + b - \frac{a^2}{2} + \frac{a}{2}\right], \quad (20)$$

where, similarly to Ref. 3, we define $\alpha_1 \equiv \int_{-\infty}^{\infty}(1 - z\hat{\xi}_1)dz$ and $\alpha_2 \equiv \int_{-\infty}^{\infty}(-\log(1+z^2) + z\hat{\xi}_2)dz$.

The value of α_1 and α_2 could be obtained integrating Eqs. (16) and (17). However, since no straightforward analytical solution for these equations is possible, we use other simple arguments to complete the calculation. First, we notice that in the absence of the equilibrium terms (i.e., $A = a = b = 0$), the dispersion relation Eq. (20) reduces to the standard visco-resistive solution, discussed in detail in Ref. 17, which implies that the coefficient $\alpha_1 \approx 2.103$.[17] The last coefficient, $\alpha_2 \approx 5.483$ was obtained by fitting Eq. (20) to numerical solution of the system 4-5 (see Sec. IV). Note that the dispersion relation, Eq. (20), can be easily extended to include the effect of the density gradients (see the Appendix).

Apart from the value of the coefficients α_1 and α_2, Eq. (20) is identical to the dispersion relation in the resistive regime,[3] once δ_μ is replaced with δ_η. The definition of the layer width significantly affects the structure of the equation since δ depends on the growth rate γ in the resistive regime, while it does not in the visco-resistive regime. This has the important consequence that while the equilibrium terms only modify the growth rate of the mode (with respect to the standard Furth *et al.* solution of Ref. 2) in the resistive regime, they can introduce a threshold in the visco-resistive regime. Indeed, it is easily seen that for the growth rate to be positive, $\Delta' > \Delta'_{cr}$, where

$$\Delta'_{cr} \equiv -\delta_\mu\alpha_1\left[\frac{Aa}{2} + a^2\log\delta_\mu - a^2\frac{\alpha_2+\pi}{\alpha_1} + b - \frac{a^2}{2} + \frac{a}{2}\right]. \quad (21)$$

Bell shaped equilibria used for modelling baseline scenario configurations[19] give a positive Δ'_{cr} although, in principle, it is not impossible to obtain negative values for this parameter. In particular, for a given equilibrium (i.e., fixed a, b, and A), Δ'_{cr} is negative if

$$\delta_\mu > \exp\left[4.601 - \frac{b + (A+1)a/2}{a^2}\right]. \quad (22)$$

This means that, in principle, for every equilibrium configuration, it is possible to have negative Δ'_{cr}, as long as the resistivity or the Prandtl number are sufficiently large. It is interesting to notice that, if a is small and b has the right sign, the condition 22 is easily met even for small values of the dissipative parameter. In the absence of asymmetric terms (i.e., $A = a = 0$), the dispersion relation and Δ'_{cr} reduce to those obtained in Ref. 16, where the viscous threshold was first identified.

IV. NUMERICAL RESULTS

It is useful, at this point, to introduce model equilibria, which can represent realistic current density profiles and allow an estimate of Δ'_{cr}. We use here two standard classes of equilibria, given by the expression

$$J_{eq} = \frac{J_0}{(1 + 16r^{2\nu})^{1+1/\nu}}, \quad (23)$$

with $\nu = 2$ for the "rounded" profile and $\nu = 4$ for the "flattened" profile.[19] The coefficient J_0 is used to modulate the total current and to change the equilibrium parameters (i.e., Δ', A, a, and b). We choose to investigate the modes characterized by $m=2$, $n=1$ and $m=3$, $n=2$ as they are the most commonly observed experimentally. Taking $R=10$, we have used a shooting code that solves for the ideal outer solution and provides Δ' and A as an output (the other equilibrium parameters can be obtained analytically in a straightforward way). Assuming $\eta=10^{-7}$ and varying the Prandtl number, we have estimated Δ'_{cr} using Eq. (21) and the results are shown in Fig. 1. For both the "rounded" and "flattened" profiles, the (3,2) modes have a higher Δ'_{cr} than the (2,1) modes. If only the symmetric term is used to determine Δ'_{cr}, as in Ref. 16, the threshold value is underestimated by an order of magnitude (for comparison, we plot it in Fig. 1 for m = 3, n = 2, and the rounded profile).

Finally, we compare the prediction of the analytic dispersion relation with numerical solutions of the linear equations obtained with an eigenvalue code. The resolution of the simulations (1051 radial points) is such that the singular layer is well resolved as confirmed by a convergence study. The numerical solutions allowed us to identify the constant α_2 by tuning it in such a way to minimize the error between the predictions of the dispersion relation, Eq. (21), and the numerical data. Obviously, this minimization process was done considering only the data that fall within the limit of validity of the theory. In order to ensure a precise evaluation of α_2, we have generated numerical solutions in disparate equilibrium and plasma conditions [e.g., all the equilibrium parameters vary greatly between "rounded" and "flattened" profiles and between (2,1) and (3,2) modes].

In particular, we have studied the (2,1) mode for $\eta=5\times10^{-8}$ and $\eta=10^{-7}$ in both "rounded," Fig. 2, and "flattened" equilibria, Fig. 3. In both cases, the agreement between the theory and the numerical data is excellent. It should be noted that the only small disagreement is observed when P is small and, at the same time, Δ' is large (compare the circle markers, for which $P=0.001$, in the upper-right part of Figs. 2–4). This, however, is easily explained in the framework of the theory we developed, since for those points $P \lesssim Q$ (or alternatively, $\Delta' \gtrsim P/\delta_\mu$), which implies that the inertial contribution in Eq. (12), which we neglect, should be taken into account.

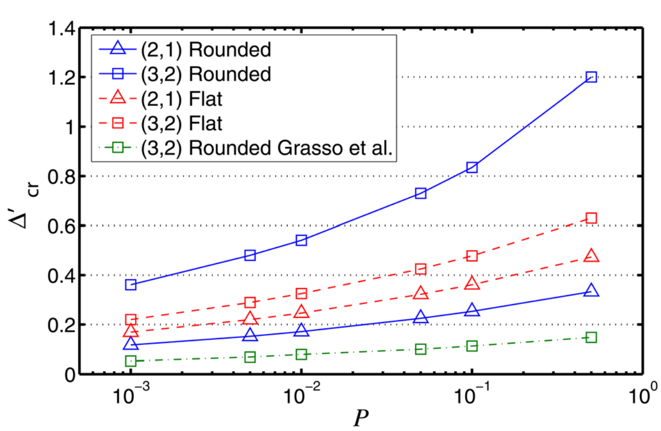

FIG. 1. (Color online) Δ'_{cr} for different modes and equilibria as a function of the Prandtl number, assuming $\eta=10^{-7}$. Δ'_{cr} calculated in Ref. 16 is shown for comparison.

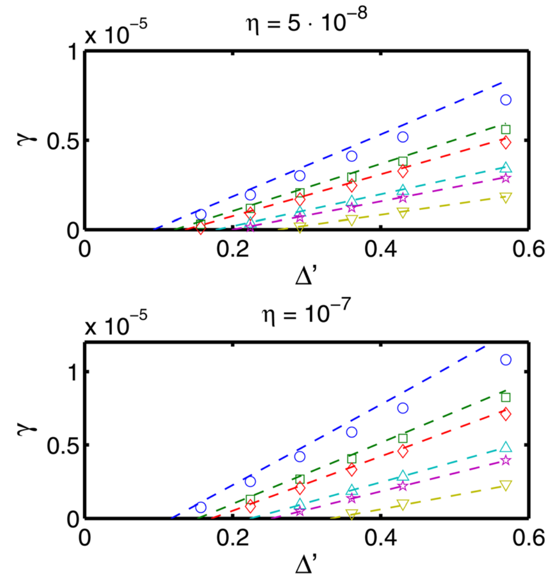

FIG. 2. (Color online) Growth rate of the (2,1) tearing mode as a function of Δ' for different values of the resistivity. A "rounded" equilibrium current density profile is assumed (Eq. (23)). The markers represent the numerical solutions for $P=0.001$ (circles), $P=0.005$ (squares), $P=0.01$ (diamonds), $P=0.05$ (triangles), $P=0.1$ (stars), and $P=0.5$ (inverted triangles). The dashed lines represent the theoretical predictions for each Prandtl number obtained with Eq. (20).

Figure 4 shows a similar very good agreement between theory and numerics for the (3,2) perturbation. For this mode, Δ'_{cr} is pushed to relatively large values, in particular in the "rounded" model. This could explain why the predictions do not exactly match the simulations for the points in the lower right part of the upper frame in Fig. 4. In these cases, the Prandtl number is approaching unity ($P=0.05$ for the triangles, $P=0.1$ for the stars, and $P=0.5$ for the inverted triangles), which implies a larger δ_μ. As a consequence, the $\hat{\psi}_1$ contribution, which scales like $\Delta'\delta_\mu$, might

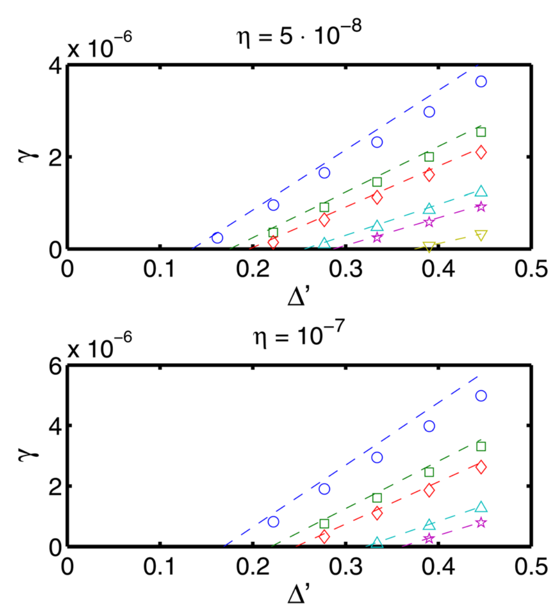

FIG. 3. (Color online) Same as Fig. 2 with "flattened" equilibrium current density.

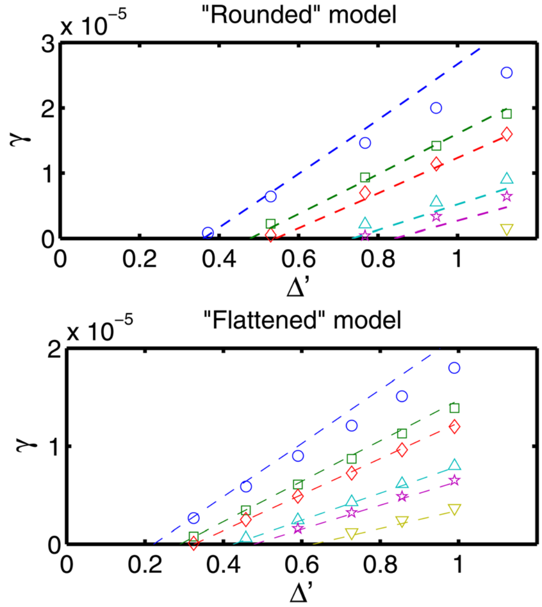

FIG. 4. (Color online) Growth rate of the (3,2) tearing mode as a function of Δ' for different equilibrium current density models. In both cases, the resistivity is $\eta = 10^{-7}$. The meaning of the symbols is the same as in Fig. 2.

become comparable to the other terms in Eq. (13). The break up of the extended constant-ψ approximation in this regime would then invalidate our treatment. Overall, once applied within its limit of validity the theory generates predictions that match the numerical data with very good precision.

V. SUMMARY AND CONCLUSIONS

We have developed a theory for the visco-resistive tearing mode in the presence of asymmetric equilibria. This regime is crucial in the definition of the linear stability of the mode as it becomes dominant close to marginality if β is low. Similarly to the symmetric case (which is a limiting case of our theory), the viscosity introduces a finite threshold for Δ', above which the perturbation becomes unstable. We have shown that the asymmetry in the equilibrium current density profile can enhance this effect and push the threshold to much higher (absolute) values. In addition, particular asymmetric equilibria can allow for negative thresholds, which implies that the tearing mode can naturally become unstable for $\Delta' < 0$. In the Appendix, the theory was extended to include the effect of finite density gradients.

The major limitation of our model is the assumption of a collisional plasma. At high temperature, relevant for modern fusion experiments, the plasma enters the semi-collisional regime and our model no longer applies. This occurs when the layer width becomes of the order of $(\omega_* \eta)^{1/2}/\rho$, where ρ is the ion sound Larmor radius or the ion Larmor radius depending on whether the ions are cold or not. In the semi-collisional regime, other threshold mechanisms were identified by several authors. These fall into two major categories: (1) the damping of the mode is associated with the coupling with the parallel ion acoustic waves[20,21] or (2) the electron temperature gradients coupled with the parallel thermal conduction stabilizes the mode.[22,23] The value of Δ'_{cr} associated to these physical effects scales like $\beta(L_s/L_n)^{1/2}/\rho$ for (1) and like $\eta_e^2\beta(L_s/L_n)^2/\rho$ for (2). Here, L_s is the magnetic shear length, L_n is the equilibrium density scale length, and $\eta_e \equiv (d \log n_{eq}/dx)/(d \log T_{e,eq}/dx)$. In both these cases, Δ'_{cr} is much larger than the one obtained with the visco-asymmetric theory presented here. Finally, it is worth noticing that in toroidal configurations, another stabilizing threshold appears[24,25] which is however smaller than (1) and (2).

As a consequence, the threshold mechanism described here is dominant only for low temperature plasmas, i.e., for sufficiently small magnetic Reynolds number. These plasma conditions can occur in magnetic reconnection experiments such as MRX (Magnetic Reconnection eXperiment) (Ref. 26) or in numerical simulations, where the resistivity is artificially increased in order to reduce the computational time.

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APPENDIX A: EFFECT OF FINITE DENSITY GRADIENTS

The visco-asymmetric dispersion relation can be straightforwardly extended to include finite density gradients. This can be achieved by replacing η with $\eta/(1 + i\omega_*/\gamma)$ in Eq. (20) (see Ref. 18 for a similar calculation with the standard constant-ψ approximation). Here, $\omega_* = (m/r_s)cTL/(v_A eBL_n)$ is the normalized electron diamagnetic frequency (c is the speed of light, e is the electron charge, and L_n is the equilibrium density scale length). It can easily be seen that for finite ω_*, the mode acquires an imaginary frequency (i.e., it rotates in the drift plane). As the plasma temperature increases, the growth rate falls below the drift frequency, $\Re(\gamma) < \omega_*$, and the instability becomes a drift-tearing mode.[27] In this case, the viscous regime described in our work is relevant only if $\omega_* \ll P^{2/3}\eta^{1/3}k^{2/3}r_s^{-2/3}$. With finite density gradients, Eq. (20) thus becomes

$$\alpha_1 \frac{\gamma r_s^2}{\eta}\left(1+i\frac{\omega_*}{\gamma}\right) = \frac{\Delta'}{\delta_\mu}\left(1+i\frac{\omega_*}{\gamma}\right)^{1/6} + \alpha_1\left[\frac{Aa}{2} + a^2\log\frac{\delta_\mu}{\left(1+i\frac{\omega_*}{\gamma}\right)^{1/6}} - a^2\frac{\alpha_2+\pi}{\alpha_1} + b - \frac{a^2}{2} + \frac{a}{2}\right]. \qquad \text{(A1)}$$

The validity of Eq. (A1) is further restricted by the condition that the broadening of the layer width is limited: $\delta_{\mu,*} \equiv \delta_\mu\left|\left(1+i\frac{\omega_*}{\gamma}\right)^{-1/6}\right| \ll 1$ (otherwise, the asymptotic matching would not be possible). The solution is, therefore, breaking up close to marginality, when the growth rate, $\Re(\gamma)$, vanishes and the mode rotates at the diamagnetic frequency: $\Im(\gamma) \approx -\omega_*$. As a consequence, in this regime, we cannot provide a simple expression for the critical Δ', the evaluation of which is left for future work.

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