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Reduced fluid-kinetic equations for low-frequency dynamics, magnetic reconnection, and electron heating in low-beta plasmas

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A minimal model for magnetic reconnection and, generally, low-frequency dynamics in low-beta plasmas is proposed. The model combines analytical and computational simplicity with physical realizability: it is a rigorous limit of gyrokinetics for plasma beta of order the electron-ion mass ratio. The model contains collisions and can be used both in the collisional and collisionless reconnection regimes. It includes gyrokinetic ions (not assumed cold) and allows for the topological rearrangement of the magnetic field lines by either resistivity or electron inertia, whichever predominates. The two-fluid dynamics are coupled to electron kinetics—electrons are not assumed isothermal and are described by a reduced drift-kinetic equation. The model, therefore allows for irreversibility and conversion of magnetic energy into electron heat via parallel phase mixing in velocity space. An analysis of the exchanges between various forms of free energy and its conversion into electron heat is provided. It is shown how all relevant linear waves and regimes of the tearing instability (collisionless, semicollisional, and fully resistive) are recovered in various limits of our model. An efficient way to simulate our equations numerically is proposed, via the Hermite representation of the velocity space. It is shown that small scales in velocity space will form, giving rise to a shallow Hermite-space spectrum, whence it is inferred that, for steady-state or sufficiently slow dynamics, the electron heating rate will remain finite in the limit of vanishing collisionality. © *2011 American Institute of Physics*. [doi:10.1063/1.3628639]

I. INTRODUCTION

Multiscale nonlinear plasma phenomena pose some of the most tantalizing and intellectually challenging theoretical problems in the field. Roughly speaking, this is because they tend to involve coupling between fluid (large) and kinetic (small) scales and so strain both our physical intuition (which in most cases is anchored in the fluid description) and analytical stamina (which tends to break down when full Vlasov-Maxwell kinetic theory appears to be the only rigorous recourse). A particularly clear example of this situation is the development of the (nonlinear) kinetic theory of magnetic reconnection.

Magnetic reconnection, the unfreezing of magnetic flux in nonideal conducting plasmas,[1,2] is one of those fundamental plasma physical processes understanding which turns out to be an essential theoretical building block in a vast range of laboratory, space, and astrophysical applications: the sawtooth crash in tokamaks,[3] solar flares,[4,5] the magnetosheath and magnetotail of the Earth,[6–8] various types of magnetohydrodynamic and plasma turbulence,[9,10] particle acceleration mechanisms,[11,12] as well as a plethora of more exotic reconnection-related phenomena believed to occur in extreme astrophysical environments[13]—solutions to these problems turn out to depend crucially on whether one thinks magnetic reconnection is fast or slow, steady or bursty and impulsive,[14] collisional or collisionless,[15,16] essentially two- or three-dimensional, mediated by resistivity,[17,18] dispersive waves,[19] instabilities of the current sheets,[20–25] turbulence,[26–29] or various combinations of these.

There appears to be a broad theoretical consensus that in many natural plasmas, reconnection cannot be fully understood in a purely magnetohydrodynamic (MHD) setting (the same is true for most other nonlinear multiscale plasma phenomena). In recent years, a dramatic increase in the raw computing power that could be brought to bear on these problems has fuelled an intense effort—and some spectacular empirical progress—in understanding kinetic reconnection using particle-in-cell (PIC) simulations,[30–35] both in two and, more recently, three dimensions. One limitation of this approach is the lack of a simple analytical framework that can help in the interpretation of numerical results. Ever since the pure MHD approach was recognized as insufficient, and even before that, attempts have continued to develop various minimal models, usually of the two-fluid kind,[36–41] with the aim of capturing the "essential physical ingredients," reflecting at any given time the evolving understanding of what those were. Finding such minimal models has been and is likely to remain necessary also because, despite the rapid expansion of brute-force simulations, the resolution available, however impressive, is in fact, never enough: the problem has at least three well separated spatial scales (global fluid, ion kinetic, and electron kinetic), as well as possibly a similar level of complexity in the kinetic phase (velocity) space, and is likely to require three dimensions in an essential way. Furthermore, the current fully kinetic PIC simulations[31] have as yet to produce a clear picture of reconnection in low-beta plasmas embedded in a strong guide field, while gyrokinetic simulations of magnetic reconnection[42–46] are in their infancy and are afflicted both by high numerical resolution requirements and lack of clarity on

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the theoretically expected outcomes (the gyrokinetic theory[47] in its general form is not much less analytically difficult that the full Vlasov-Maxwell kinetics, although numerically it is, of course, a radical simplification).

There is a good reason to believe that a physically realizable model set of equations for weakly collisional reconnection, or nonlinear plasma dynamics generally, however simplified, cannot in general be fluid. Like most nonlinear phenomena, magnetic reconnection can and, indeed, should be thought about in terms of energy conversion, namely, conversion of magnetic energy associated with the reconnecting large-scale configuration into other forms of plasma energy and ultimately, particle heat. It is when the energy is dissipated into heat that this conversion becomes irreversible. However, when collisions are weak, Ohmic (resistive) heating is not an important process, so most purely fluid models of collisionless reconnection turn out to be Hamiltonian:[36,37,40,41] reconnection converts magnetic energy into other forms of "fluid" energy (e.g., kinetic energy of the mean electron flows), the entropy of the system does not increase, everything is in principle reversible, and there is no heating. This is somewhat similar to inviscid dynamics in a neutral fluid. In fact, just like in neutral fluids, nonlinear dynamics in plasmas generically trigger formation of small scales, so even small dissipation coefficients become non-negligible because they multiply large gradients (which is why dynamics in fluids with small viscosity are not everywhere inviscid, boundary layers form). In weakly collisional plasmas, the small-scale structure arises in phase space (i.e., both in position and particle velocities) via linear and nonlinear phase mixing processes (see Sec. V D). Once large velocity-space gradients are present, even weak collisions are sufficient to dissipate energy and cause heating. Formally, this can be understood by tracing the evolution and flows of free energy—the quadratic invariant whose transfer to small scales and eventual thermalization is the central process in the non-equilibrium thermodynamics of kinetic plasmas (in the context of kinetic and gyrokinetic turbulence, this is explained in Refs. 48 and 49, where further references are provided; in application to magnetic reconnection, we expand on this topic in some detail in Sec. IV). Thus, we believe there is a need for a minimal model of weakly collisional reconnection that is *not* fully conservative, because exact conservation properties impose constraints on the phase space of the system[50,51] the breaking of which is not just non-negligible but is in fact likely to be physically essential in real plasmas.[108] While numerical evidence is perhaps not fully conclusive on this subject, a few recent studies have, explicitly or implicitly, stressed the importance of electron heating in the nonlinear regime of kinetic reconnection.[35,45,52]

In this paper, we take the view that a minimal model that is as fluid-like as possible is clearly desirable, but *ad hoc* closure approximations are dangerous even if they appear to be physically motivated. We would like, therefore, to have a model that combines analytical and computational simplicity with physical realizability, i.e., constitutes a rigorously correct approximation of the kinetic system in some well defined physical limit. It turns out such a model can indeed be constructed and is a simple generalization of an existing two-fluid model,[37] although it is not a fluid model in that it does retain kinetic electrons: the equations are the continuity equation, the "gyrokinetic Poisson equation" for the ions, the generalized Ohm's law and a version of the drift-kinetic equation for electrons; the latter is coupled to the fluid variables via the gradient of parallel electron temperature (energetically, via work done by the parallel electron pressure gradient). The formal limit in which this model is derived is a combination of the gyrokinetic regime (strong guide field, frequencies below ion cyclotron, strong anisotropy $k_\parallel \ll k_\perp$, small-amplitude fluctuations) with a low-beta expansion (plasma beta of order electron-ion mass ratio)—this is explained in detail in Sec. II B. A very close precursor of our approach in the existing literature is the fluid-kinetic model proposed by de Blank,[53,54] which couples a two-fluid model to a simplified electron kinetic equation, although he does not give a rigorous asymptotic ordering under which his model holds and also restricts his attention to the exactly 2D, exactly collisionless case, which means that his model is Hamiltonian and has an infinite number of Lagrangian conserved quantities[55,56] (see Appendix A).

Let us discuss the basic physical ingredients that are retained in our approach. Despite the simplicity of our equations (summarized in Sec. III E), we believe that very little is lost of what we consider the *sine qua non* of kinetic reconnection.

Three scales. The problem is fundamentally three-scale: the model has to allow for a reconnecting configuration on fluid scales, dispersive effects at the ion scales and the flux unfreezing by a collisionless mechanism associated with electron microphysics and so residing at electron scales. In our case, the ion scales are the ion sound and Larmor radii (the ions are gyrokinetic and not assumed cold; see Sec. III C). The flux unfreezing is effected by electron inertia—see Sec. III B; we will sacrifice the electron-Larmor-radius effects.

Collisions. It is desirable for a good model to contain a smooth transition from collisionless to semicollisional to fully collisional (resistive MHD) regime—both because this provides a way to benchmark against situations that are better understood and because theoretically it is possible that many natural systems teeter at the boundary between the collisionless and collisional regimes.[15,16] The transition between the two has been the focus of several recent studies, both experimental[57] and theoretical,[34,35,46,58] but remains poorly understood. In our model, electron-ion collisions are retained (see Sec. III D) and so both Ohmic resistivity and (parallel) electron heat conduction (in the semicollisional limit; see Sec. V C) is recoverable (but not ion or electron viscosity, a limitation that will be discussed further in Sec. VI).

Free-energy flows, Landau damping and electron heating. Finally, as we explained above, the model provides an electron heating channel, operative even with very weak collisions, in the form of the coupling of the generalized Ohm's law to the electron kinetic equation with a collision operator retained. This means that the electron Landau damping (whose essential byproduct is parallel phase mixing) is included and free energy can be converted from various fluid forms into the electron entropy and thence to heat (see

Sec. IV B). Note that neither ion Landau damping nor the nonlinear perpendicular phase mixing of either species[48,49,59] survive in our model.

The rest of the paper is organized as follows. In Sec. II, we review the gyrokinetic system, which is our starting point (Sec. II A), and introduce the set of ordering assumptions that encode the physical limit in which our equations hold rigorously (Sec. II B). In Sec. III, we derive the equations themselves (they are summarized in Sec. III E). In Sec. IV, we work out the energetics of these equations: the various forms the free energy takes (Sec. IV A), how it is exchanged between fields (Sec. IV B) and what that implies about irreversibility, thermalization (dissipation), and electron heating (Sec. IV C). In Sec. V, we introduce a spectral representation of the electron kinetics in terms of Hermite polynomials (using a modified Lenard-Bernstein operator for electron collisions, introduced in Sec. III D). This provides what appears to be both a remarkably simple computational approach and an intuitively appealing physical interpretation of velocity-space dynamics and electron heating as a cascade in Hermite space (Sec. V D). We are led to the conclusion that the electron heating rate should remain finite in the limit of positive but vanishing electron collision frequency—except for fast-growing solutions like the tearing mode (Sec. V E). We also derive the semicollisional limit of our equations[60] in Sec. V C. Section VI contains the concluding discussion. The paper is supplemented with two technical appendices: on two-dimensional invariants of our system (Sec. A) and on the linear theory [gyrokinetic electron-Landau-damped Alfvén waves,[61] collisional (resistive MHD),[62–65] semicollisional[66–70] and collisionless[65–67,71–76] tearing modes].

We have adopted a rather gradual, step-by-step approach to the derivation of all results. An impatient reader, or one already familiar with most of the (standard) analytical machinery deployed here, can gain a basic idea of the main results and conclusions by looking at Secs. III E, IV C, V E, and VI.

II. PRELIMINARIES

A. Gyrokinetics in a slab

The starting point for our derivation is the gyrokinetic description of magnetized plasma,[47] which is appropriate for low-frequency ($\omega \ll \Omega_s$) anisotropic ($k_\parallel \ll k_\perp$) fluctuations in the presence of a mean magnetic field. The simplest case, which is all we will require here, is that of a static uniform equilibrium with zero electric field and a constant straight magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, whose direction defines the z axis. The subscripts $\perp$ and $\parallel$ will always refer to directions with respect to this equilibrium ("guide") magnetic field. The gyrokinetic equations are derived by performing an expansion of the Vlasov-Landau equation in the small parameter $\epsilon = k_\parallel/k_\perp$, and averaging out the particle Larmor motion, i.e., all frequencies greater than the cyclotron frequency $\Omega_s = q_s B_0/M_s c$, where $s = i, e$ is the species index, q_s is particle charge, m_s is particle mass, and c is the speed of light. For a detailed derivation of homogeneous gyrokinetics in a slab, the reader may consult Ref. 61.

Let us summarize the resulting equations. The distribution function up to first order in ϵ is

$$f_s(\mathbf{r},\mathbf{v},t) = F_{0s} - \frac{q_s\varphi(\mathbf{r},t)}{T_{0s}}F_{0s} + h_s(\mathbf{R}_s, v_\perp, v_\parallel, t), \quad (1)$$

where the zeroth-order distribution

$$F_{0s}(v) = \frac{n_{0s}}{\left(\pi v_{\mathrm{ths}}^2\right)^{3/2}}\exp\left[-\frac{v_\parallel^2 + v_\perp^2}{v_{\mathrm{ths}}^2}\right] \quad (2)$$

is a Maxwellian with uniform density n_{0s} and temperature T_{0s}, $v_{\mathrm{ths}} = (2T_{0s}/m_s)^{1/2}$ is the thermal speed, $q_s\varphi/T_{0s} = O(\epsilon)$ is the Boltzmann response containing the electrostatic potential φ, and h_s is the gyrocenter distribution function, also $O(\epsilon)$. Spatially, h_s is defined as a function of the gyrocenter, or guiding-center, coordinate

$$\mathbf{R}_s = \mathbf{r} + \frac{\mathbf{v}_\perp \times \hat{\mathbf{z}}}{\Omega_s} \quad (3)$$

and satisfies the gyrokinetic equation

$$\frac{\partial h_s}{\partial t} + v_\parallel\frac{\partial h_s}{\partial z} + \frac{c}{B_0}\left\{\langle\chi\rangle_{\mathbf{R}_s}, h_s\right\} = \frac{q_s F_{0s}}{T_{0s}}\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial t} + \left(\frac{\partial h_s}{\partial t}\right)_c, \quad (4)$$

where $\chi(\mathbf{r}, \mathbf{v}, t) = \varphi - \mathbf{v}\cdot\mathbf{A}/c$ is the gyrokinetic potential (containing the information about the electromagnetic field in the form of the scalar potential φ and vector potential $\mathbf{A}$ of the perturbed magnetic field, $\delta\mathbf{B} = \nabla\times\mathbf{A}$),

$$\left\{\langle\chi\rangle_{\mathbf{R}_s}, h_s\right\} = \hat{\mathbf{z}}\cdot\left(\frac{\partial\langle\chi\rangle_{\mathbf{R}_s}}{\partial\mathbf{R}_s}\times\frac{\partial h_s}{\partial\mathbf{R}_s}\right) \quad (5)$$

is the Poisson brackets, $(\partial h_s/\partial t)_c$ is the (gyroaveraged) collision operator, and

$$\langle\chi(\mathbf{r},\mathbf{v},t)\rangle_{\mathbf{R}_s} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta\,\chi\left(t, \mathbf{R}_s - \frac{\mathbf{v}_\perp\times\hat{\mathbf{z}}}{\Omega_s}, \mathbf{v}\right) \quad (6)$$

is the average of χ at constant $\mathbf{R}_s$ over the gyroangles ϑ. In Fourier space, this gyroaveraging operation takes a simple mathematical form in terms of Bessel functions J_0 and J_1, so the Fourier transform of $\langle\chi(\mathbf{r},\mathbf{v},t)\rangle_{\mathbf{R}_s}$ with respect to $\mathbf{R}_s$ can be written as follows:

$$\langle\chi\rangle_{\mathbf{R}_s,\mathbf{k}} = J_0(a_s)\left(\varphi_\mathbf{k} - \frac{v_\parallel A_{\parallel\mathbf{k}}}{c}\right) + \frac{T_{0s}}{q_s}\frac{2v_\perp^2}{v_{\mathrm{ths}}^2}\frac{J_1(a_s)}{a_s}\frac{\delta B_{\parallel\mathbf{k}}}{B_0}, \quad (7)$$

where $a_s = k_\perp v_\perp/\Omega_s$, and $\varphi_\mathbf{k}$, $A_{\parallel\mathbf{k}}$, and $\delta B_{\parallel\mathbf{k}}$ are Fourier transforms (with respect to $\mathbf{r}$) of the scalar potential, parallel component of the vector potential and parallel component of the perturbed magnetic field, respectively. These fields are determined via Maxwell's equations, namely, the quasineutrality and Ampère's law, where particle densities and currents are calculated from the gyrocenter distribution h_s. These equations will be introduced in Secs. III A and III C, where we will need them to compute electron flow velocity and density perturbation.

B. Low-beta ordering

We would like to derive a minimal model suitable for an analytical description of magnetic reconnection in the presence of a mean field, i.e., reconnection of antiparallel perturbations $\delta\mathbf{B}_\perp = -\hat{\mathbf{z}} \times \nabla A_\parallel$. Since we wish to have a model that describes a real physical situation, we cannot resort to writing *ad hoc* fluid equations. Instead, our model will take the form of an asymptotic expansion of the gyrokinetic equations under an appropriate physically motivated ordering of all spatial and time scales and of the perturbation amplitudes. In devising our ordering, we are guided by what is known or expected about the physical effects that are essential in any description of a kinetic reconnection process.

1. Spatial scales

Firstly, in the presence of a strong guide field, electron inertia is expected to be a key mechanism for breaking the magnetic field lines (flux unfreezing) in collisionless or weakly collisional plasma.[62,77–80] Thus, we order

$$k_\perp d_e \sim 1, \tag{8}$$

where $d_e = c/\omega_{pe} = \rho_e/\sqrt{\beta_e}$ is the electron inertial scale and, to fix standard notation, $\omega_{pe} = (4\pi n_{0e} e^2/m_e)^{1/2}$ is the electron plasma frequency, $e = |q_e|$ the elementary charge, $\rho_e = v_{\mathrm{the}}/\Omega_e$ the electron Larmor radius, and $\beta_e = 8\pi n_{0e} T_{0e}/B_0^2$ the electron beta.

Secondly, a key feature of kinetic reconnection is a double layer resulting from the decoupling of the electrons from the ions at the ion sound scale.[1] To retain this effect, we order

$$k_\perp \rho_s \sim 1, \tag{9}$$

where $\rho_s = \rho_i\sqrt{Z/2\tau}$ is the ion sound radius (subscript s for "sound" not to be confused with the species index!), $\rho_i = v_{\mathrm{thi}}/\Omega_i$ is the ion Larmor radius, $Z = q_i/e$ and $\tau = T_{0i}/T_{0e}$. We will consider the temperature ratio to be order unity, so Eq. (9) immediately implies

$$\tau \sim 1, \quad k_\perp \rho_i \sim \left(\frac{\tau}{Z}\right)^{1/2} \sim 1. \tag{10}$$

Thus, we retain the ion finite Larmor radius (FLR) along with the ion sound scale. However, a further simplifying option remains open: assuming cold ions, $\tau \ll 1$, we can eliminate the ion FLR effects. It is with a view to this possibility that we will carry the τ dependence in all our orderings discussed below.

In order for Eqs. (8) and (9) to be consistent, we must have, within our ordering, $d_e \sim \rho_s$ (this does not mean that these scales must be similar, just that we are not hard-wiring into our model a disparity between them). This implies

$$\frac{d_e}{\rho_s} = \sqrt{2Z}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{1}{\sqrt{\beta_e}} \sim 1. \tag{11}$$

To achieve this, we order

$$\beta_e \sim \frac{Z m_e}{m_i} \ll 1. \tag{12}$$

Thus, we are restricting our consideration to low-beta plasmas and allowing both species to have finite temperature. The ordering (12) is appropriate, for example, for the solar corona[16] and low-beta laboratory experiments such as the LArge Plasma Device at UCLA.[81] In modern tokamaks, values of $\beta_e \sim 10^{-3}$ can occur in the edge pedestal region in the H-mode regime.[82]

Note that in the gyrokinetic approximation, β_e is considered order unity with respect to the ordering of all quantities in powers of $\epsilon = k_\parallel/k_\perp$; we will treat our low-beta expansion as subsidiary to the gyrokinetic ϵ expansion.

Now, using Eq. (8), we conclude

$$k_\perp \rho_e \sim \sqrt{\beta_e} \ll 1, \tag{13}$$

which will allow us to neglect the electron FLR effects and derive an "almost fluid" set of equations for the electrons.

2. Time scales and perturbation amplitudes

Far from the reconnecting region, the plasma mass flow is ordered with the $\mathbf{E}\times\mathbf{B}$ drift velocity. Thus, we order the fundamental time scale on which we allow our fields to vary in such a way that the characteristic frequency is that associated with the $\mathbf{E}\times\mathbf{B}$ velocity $u_\perp$:

$$\omega \sim k_\perp u_\perp \sim k_\perp^2 \frac{c\varphi}{B_0}. \tag{14}$$

Note that the gyrokinetic approximation requires $\omega \ll \Omega_{i,e}$.

Since it is an essential feature of our model to take into account electron kinetics, we must allow the parallel streaming frequency of the electrons to be the same order as the rate at which our fields vary, namely

$$\omega \sim k_\parallel v_{\mathrm{the}}. \tag{15}$$

The requirement that Eqs. (14) and (15) should be consistent with each other imposes an ordering on the size of the scalar potential:

$$\frac{e\varphi}{T_{0e}} \sim \frac{k_\parallel}{k_\perp}\frac{1}{k_\perp\rho_e} \sim \frac{\epsilon}{\sqrt{\beta_e}}, \tag{16}$$

where we used Eq. (13). Note that the appearance of the gyrokinetic expansion parameter ϵ in the ordering of the perturbation amplitudes confirms that the use of the gyrokinetic approximation is appropriate in the physical circumstances we are considering.

We further require that the density perturbations are of the same order as the electrostatic perturbations given by Eq. (16):

$$\frac{\delta n_e}{n_{0e}} \sim \frac{Z}{\tau}\frac{e\varphi}{T_{0e}} \sim \frac{Z}{\tau}\frac{\epsilon}{\sqrt{\beta_e}} \tag{17}$$

(the factor of Z/τ will emerge in Sec. III C and is kept for book-keeping purposes). Physically this follows from the requirement that the physics associated with the ion sound scale is retained [Eq. (9)].

3. Alfvénic perturbations

We will now order the magnetic perturbations. Let us observe that the ordering (12) implies that the electron thermal speed is comparable to the Alfvén speed $v_A = B_0/\sqrt{4\pi m_i n_{0i}}$: since $n_{0i} = n_{0e}/Z$ (quasineutrality), we have

$$\frac{v_{\mathrm{the}}}{v_A} = \left(\frac{\beta_e}{Z}\frac{m_i}{m_e}\right)^{1/2} \sim 1. \tag{18}$$

Therefore,

$$\omega \sim k_\parallel v_A, \tag{19}$$

i.e., Alfvén waves can propagate along the guide field with the same characteristic frequency as the electrons stream and plasma flows. Note that in view of Eqs. (9) and (10), this ordering holds both for the magnetohydrodynamic Alfvén waves $(\omega = k_\parallel v_A)$, and for the kinetic Alfvén waves $(\omega \sim k_\parallel v_A k_\perp \rho_s)$.

Stipulating that Alfvénic perturbations should be accommodated by our ordering (which is equivalent to demanding that the Lorentz force is nonnegligible—an essential ingredient in a reconnecting system), we deduce the ordering for the perpendicular perturbed magnetic field $\delta\mathbf{B}_\perp = -\hat{\mathbf{z}} \times \nabla_\perp A_\parallel$:

$$\frac{\delta B_\perp}{B_0} \sim \frac{u_\perp}{v_A} \sim \epsilon k_\perp \rho_i \left(\frac{Z}{\tau}\right)^{1/2} \sim \epsilon, \tag{20}$$

where we have used Eqs. (16) and (10).

Note that, since $\epsilon \sim k_\parallel/k_\perp$, this implies $k_\perp \delta B_\perp \sim k_\parallel B_0$, i.e., the spatial variation of all quantities along the exact magnetic field contains comparable contributions from their variation along both the mean and perturbed fields—a general property of gyrokinetic perturbations. This is just what is needed for reconnection problems with a guide field because Alfvénic dynamics with respect to the perturbed field must be allowed.

Finally, from the perpendicular component of Ampère's law [see, e.g., Eq. (120) of Ref. 49],

$$\frac{\delta B_\parallel}{B_0} \sim \beta_e \frac{e\varphi}{T_{0e}} \sim \epsilon\sqrt{\beta_e}, \tag{21}$$

where Eq. (16) has been used. This will cause the parallel perturbations of the magnetic field (i.e., perturbations of the field strength) to fall out of our final set of equations. Indeed, it is a well known fact that in the low-beta ordering such perturbations tend to be negligible.

This completes the ordering of the perturbation amplitudes.

4. Resistivity and collisions

While our main focus is on collisionless reconnection, we would like to make contact with the collisional limit, in which the magnetic flux unfreezing and magnetic energy release are accomplished by Ohmic resistivity.[83] We can retain resistivity by ordering the electron-ion (and consequently electron-electron) collision frequency as comparable to the characteristic frequency of all the other processes that we are taking into account:

$$\nu_{ei} = Z\nu_{ee} \sim \omega. \tag{22}$$

Since the Ohmic magnetic diffusivity (often colloquially called resistivity) is $\eta \sim \nu_{ei} d_e^2$, the above ordering means that the diffusive effects are retained in our ordering: indeed, using Eqs. (8) and (22),

$$\eta k_\perp^2 \sim \nu_{ei} k_\perp^2 d_e^2 \sim \nu_{ei} \sim \omega. \tag{23}$$

The resistivity can later be neglected in a subsidiary collisionless expansion, but we consider it useful to have a model in which a smooth transition from collisional to collisionless regime is possible (cf. Refs. 34, 35, 46, 57 and ideas on the marginal collisional-collisionless reconnection regime[15,16,58]).

Our ordering of the electron collision immediately implies an ordering of the ion collisions:[84]

$$\nu_{ii} = \frac{Z^2}{\tau^{3/2}}\left(\frac{m_e}{m_i}\right)^{1/2}\nu_{ei} \sim \left(\frac{Z}{\tau}\right)^{3/2}\sqrt{\beta_e}\omega, \tag{24}$$

$$\nu_{ie} = \frac{Zm_e}{m_i}\nu_{ei} \sim \beta_e \omega, \tag{25}$$

where we have used Eqs. (12) and (22). Thus, we are effectively assuming ions to be collisionless (no ion viscosity).

Finally, it is perhaps useful to note the ordering of the particle mean free path: using Eqs. (15) and (22), we have

$$k_\parallel \lambda_{\mathrm{mfp}e} = \left(\frac{Z}{\tau}\right)^2 k_\parallel \lambda_{\mathrm{mfp}i} = \frac{k_\parallel v_{\mathrm{the}}}{\nu_{ei}} \sim 1. \tag{26}$$

III. DERIVATION OF THE EQUATIONS

We are now ready to apply our ordering and derive from the gyrokinetic system (Sec. II A) a reduced system of equations describing gyrokinetic ions and drift-kinetic electrons in a low-beta weakly collisional plasma. An impatient reader can skip to Sec. III E.

A. Electrons

In Eq. (4) for electrons ($s = e$), we retain only the lowest order in the expansion with respect to β_e. In order to do this, we note first that the Bessel functions in the expression for $\langle\chi\rangle_{\mathbf{R}_e}$ [Eq. (7)] can be expanded in small argument because $a_e \sim k_\perp \rho_e \sim \sqrt{\beta_e} \ll 1$ [Eq. (13)]:

$$J_0(a_e) \simeq \frac{2J_1(a_e)}{a_e} = 1 + O(a_e^2). \tag{27}$$

Using Eqs. (16) and (20), we find that the φ and $A_\parallel$ terms in Eq. (7) are the same order,

$$\frac{v_\parallel A_\parallel}{c} \sim \frac{v_{\mathrm{the}} B_0}{ck_\perp}\frac{\delta B_\perp}{B_0} \sim \frac{T_{0e}}{e}\frac{\epsilon}{k_\perp \rho_e} \sim \varphi, \tag{28}$$

while, according to Eq. (21), the $\delta B_\parallel$ term is one order of β_e smaller. Thus,

$$\langle \chi \rangle_{\mathbf{R}_e} = \left(\varphi - \frac{v_\parallel A_\parallel}{c} \right) [1 + O(\beta_e)]. \tag{29}$$

This allows us to write Eq. (4) as follows, up to corrections of order $O(\beta_e)$,

$$\frac{dh_e}{dt} + v_\parallel \hat{\mathbf{b}} \cdot \nabla h_e = -\frac{eF_{0e}}{T_{0e}} \frac{\partial}{\partial t} \left(\varphi - \frac{v_\parallel A_\parallel}{c} \right) + \left(\frac{\partial h_e}{\partial t} \right)_c, \tag{30}$$

where we have introduced a "convective" time derivative incorporating the $\mathbf{E} \times \mathbf{B}$ motion and the parallel spatial derivative along the perturbed field line:

$$\frac{dh_e}{dt} = \frac{\partial h_e}{\partial t} + \frac{c}{B_0} \{ \varphi, h_e \}, \tag{31}$$

$$\hat{\mathbf{b}} \cdot \nabla h_e = \frac{\partial h_e}{\partial z} - \frac{1}{B_0} \{ A_\parallel, h_e \}. \tag{32}$$

Note that all terms in Eq. (30) are comparable under ordering because of the assumptions we made about the time scales for the $\mathbf{E} \times \mathbf{B}$ flows [Eq. (14)], electron streaming [Eq. (15)], collisions [Eq. (22)] and perturbation amplitudes [Eqs. (16) and (20)].

It is now convenient formally to split the electron distribution function in such a way as to separate the inhomogeneous solution arising from the first term on the right-hand side of Eq. (30) as well as the density and parallel velocity moments:

$$h_e = \left(-\frac{e\varphi}{T_{0e}} + \frac{\delta n_e}{n_{0e}} + \frac{2v_\parallel u_{\parallel e}}{v_{\mathrm{the}}^2} \right) F_{0e} + g_e, \tag{33}$$

where by definition

$$\frac{\delta n_e}{n_{0e}} = \frac{1}{n_{0e}} \int d^3\mathbf{v} \delta f_e, \tag{34}$$

$$u_{\parallel e} = \frac{1}{n_{0e}} \int d^3\mathbf{v} v_\parallel \delta f_e, \tag{35}$$

and $\delta f_e = h_e + e\varphi F_{0e}/T_{0e}$ is the total perturbed electron distribution function [see Eq. (1)]. Since the first term in Eq. (33) cancels the Boltzmann part in δf_e, the above definitions of the density and parallel electron flow velocity are consistent provided we demand that

$$\int d^3\mathbf{v} \begin{pmatrix} 1 \\ v_\parallel \end{pmatrix} g_e = 0, \tag{36}$$

i.e., the "reduced" electron distribution function g_e contains all higher moments of the electron distribution function, but not density or parallel flow.

The perturbed density will be calculated in Sec. III C from the quasineutrality condition. The parallel electron velocity $u_{\parallel e}$ is related to $A_\parallel$ and to the parallel ion velocity $u_{\parallel i}$ via the parallel component of Ampère's law:

$$\hat{\mathbf{z}} \cdot (\nabla_\perp \times \delta \mathbf{B}_\perp) = -\nabla_\perp^2 A_\parallel = \frac{4\pi}{c} j_\parallel = \frac{4\pi e n_{0e}}{c} (u_{\parallel i} - u_{\parallel e}). \tag{37}$$

This can be easily manipulated into

$$u_{\parallel e} = \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel + u_{\parallel i}. \tag{38}$$

We now substitute Eq. (33) into Eq. (30) and take moments of the resulting equation. The zeroth moment is, after using Eq. (38),

$$\frac{d}{dt} \frac{\delta n_e}{n_{0e}} + \hat{\mathbf{b}} \cdot \nabla u_{\parallel i} = -\hat{\mathbf{b}} \cdot \nabla \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel. \tag{39}$$

The first $v_\parallel$ moment of Eq. (30) is, again using Eq. (38) and dividing through by en_{0e}/cm_e,

$$\frac{d}{dt} (A_\parallel - d_e^2 \nabla_\perp^2 A_\parallel) = -c \frac{\partial \varphi}{\partial z} + \frac{cT_{0e}}{e} \hat{\mathbf{b}} \cdot \nabla \left(\frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\parallel e}}{T_{0e}} \right) - \frac{cm_e}{en_{0e}} \int d^3\mathbf{v} v_\parallel \left(\frac{\partial h_e}{\partial t} \right)_c + \frac{cm_e}{e} \frac{du_{\parallel i}}{dt}, \tag{40}$$

where the parallel electron temperature perturbation has been introduced as a shorthand for the $v_\parallel^2$ moment of g_e:

$$\frac{\delta T_{\parallel e}}{T_{0e}} = \frac{1}{n_{0e}} \int d^3\mathbf{v} \frac{2v_\parallel^2}{v_{\mathrm{the}}^2} g_e. \tag{41}$$

Finally, an equation for g_e is obtained from Eq. (30) by subtracting from it Eq. (39) multiplied by F_{0e} and substituting $\partial A_\parallel / \partial t$ calculated using Eq. (40). After a few lines of straightforward algebra, this gives

$$\frac{dg_e}{dt} + v_\parallel \hat{\mathbf{b}} \cdot \nabla \left(g_e - \frac{\delta T_{\parallel e}}{T_{0e}} F_{0e} \right) - C[g_e] = \left(1 - \frac{2v_\parallel^2}{v_{\mathrm{the}}^2} \right) F_{0e} \hat{\mathbf{b}} \cdot \nabla \left(\frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel + u_{\parallel i} \right). \tag{42}$$

where the collisional terms have been assembled together: by definition,

$$C[g_e] = \left(\frac{\partial h_e}{\partial t} \right)_c - \frac{2v_\parallel F_{0e}}{v_{\mathrm{the}}^2 n_{0e}} \int d^3\mathbf{v}' v_\parallel' \left(\frac{\partial h_e}{\partial t} \right)_c. \tag{43}$$

It is easy to verify that Eq. (42) respects the constraint on g_e given by Eq. (36). We stress that Eq. (42) is not homogeneous in g_e, so that g_e cannot be consistently neglected (cf. Ref. 37).

B. Flux conservation

Equation (40) can easily be rearranged into the following form:

$$\frac{\partial A_\parallel}{\partial t} = -c \hat{\mathbf{b}} \cdot \nabla \tilde{\varphi} + \eta \nabla_\perp^2 A_\parallel + \frac{d}{dt} d_e^2 \nabla_\perp^2 A_\parallel, \tag{44}$$

where

$$\tilde{\varphi} \equiv \varphi - \frac{T_{0e}}{e} \left(\frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\parallel e}}{T_{0e}} \right). \tag{45}$$

Recalling that $\delta \mathbf{B}_\perp = -\hat{\mathbf{z}} \times \nabla_\perp A_\parallel$ and $\mathbf{B} = B_0 \hat{\mathbf{z}} + \delta \mathbf{B}_\perp$, we immediately infer from Eq. (44) that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_{\rm eff} \times \mathbf{B}) + \eta \nabla_\perp^2 \mathbf{B} - \hat{\mathbf{z}} \times \nabla_\perp \frac{d}{dt} d_e^2 \nabla_\perp^2 A_\parallel, \quad (46)$$

where $\mathbf{u}_{\rm eff} = \hat{\mathbf{z}} \times \nabla_\perp c\tilde{\varphi}/B_0$ is the effective velocity, into which, as the above equation demonstrates, the field lines are frozen except for the resistive and electron-inertia effects. This is an application to our equations of the more general flux-conservation argument due to Cowley.[85]

An important conclusion is that the kinetic electron effects, which enter via $\delta T_{\parallel e}$ determined from Eqs. (41) and (42), do not unfreeze flux.

C. Ions

Equations (39)–(42) are four equations that involve six quantities δn_e, φ, $A_\parallel$, $u_{\parallel i}$, $\delta T_{\parallel e}$, and g_e. In order to have a closed system, we need find additional equations for two of these quantities. The ion gyrokinetics will give us this additional information: namely, we will determine the ion parallel flow velocity $u_{\parallel i}$, which turns out to vanish to lowest order, and $\delta n_e/n_{0e}$, which, because of the quasineutrality, is the same as the ion density perturbation $\delta n_i/n_{0i}$.

As we explained in Sec. II B, the ions remain fully gyrokinetic under our ordering, i.e., the Bessel functions in the expression for $\langle \chi \rangle_{\mathbf{R}_i}$ [Eq. (7)] cannot be expanded in small argument because $a_i \sim k_\perp \rho_i$ is finite [Eq. (10)]. However, a significant simplification of Eq. (7) for ions is possible because, as in the case of the electrons, the $\delta B_\parallel$ term is an order of β_e smaller than the φ term [Eq. (21)] and also because, unlike for the electrons, the $A_\parallel$ is also small: indeed, noticing that $v_{\rm thi}/v_{\rm the} = (\tau m_e/m_i)^{1/2}$ and comparing with Eq. (28), we find

$$\frac{v_\parallel A_\parallel}{c} \sim \frac{v_{\rm thi} B_0}{c k_\perp} \frac{\delta B_\perp}{B_0} \sim \left(\tau \frac{m_e}{m_i}\right)^{1/2} \varphi \sim \frac{\tau}{Z} \sqrt{\beta_e} \varphi. \quad (47)$$

Thus, we have for the ions $\chi = \varphi$, or

$$\langle \chi \rangle_{\mathbf{R}_i, \mathbf{k}} = J_0(a_i) \varphi_{\mathbf{k}}. \quad (48)$$

Because of the way we ordered the electron streaming frequency [Eq. (15)] and the electron collisions [Eq. (22)], the ion streaming frequency

$$k_\parallel v_{\rm thi} = \left(\tau \frac{m_e}{m_i}\right)^{1/2} k_\parallel v_{\rm the} \sim \frac{\tau}{Z} \sqrt{\beta_e} \omega \quad (49)$$

and the ion collisions [Eqs. (24) and (25)] are small and so the corresponding terms in Eq. (4) for ions ($s = i$) are negligible to lowest order. What remains is the following rather simple equation for the ions, devoid of kinetic effects except for the ion FLR

$$\frac{\partial g_i}{\partial t} + \frac{c}{B_0} \left\{ \langle \varphi \rangle_{\mathbf{R}_i}, g_i \right\} = 0, \quad (50)$$

where we have introduced a new function

$$g_i = h_i - \frac{Ze F_{0i}}{T_{0i}} \langle \varphi \rangle_{\mathbf{R}_i}. \quad (51)$$

Equation (50) is homogeneous and has one very straightforward solution:

$$g_i = 0. \quad (52)$$

Thus, the ion response under our ordering is essentially electrostatic and the perturbed ion distribution function is [see Eq. (1)]

$$\delta f_i = \frac{Ze F_{0i}}{T_{0i}} \left(\langle \varphi \rangle_{\mathbf{R}_i} - \varphi \right). \quad (53)$$

Note that the first term here is a function of the gyrocenter variable $\mathbf{R}_i$ while the second is a function of the position variable $\mathbf{r}$. When computing the ion density and flow velocity, we must integrate δf_i over velocities while keeping $\mathbf{r}$ constant, which means that terms dependent on $\mathbf{R}_i$ must be gyroaveraged at constant $\mathbf{r}$. Just like the gyroaveraging at constant $\mathbf{R}_i$, this can be expressed in Fourier space as a multiplication by the Bessel function $J_0(a_i)$. Thus, we calculate

$$\begin{aligned} \frac{\delta n_i}{n_{0i}} &= \frac{1}{n_{0i}} \int d^3\mathbf{v} \langle \delta f_i \rangle_{\mathbf{r}} \\ &= \frac{Ze}{T_{0i}} \left(\frac{1}{n_{0i}} \int d^3\mathbf{v} \langle \langle \varphi \rangle_{\mathbf{R}_i} \rangle_{\mathbf{r}} F_{0i} - \varphi \right) \\ &= -(1 - \hat{\Gamma}_0) \frac{Ze\varphi}{T_{0i}}, \end{aligned} \quad (54)$$

where $\hat{\Gamma}_0$ is the operator that is the inverse Fourier transform of

$$\Gamma_0(\alpha_i) = \frac{1}{n_{0i}} \int d^3\mathbf{v} [J_0(a_i)]^2 F_{0i} = I_0(\alpha_i) e^{-\alpha_i}, \quad (55)$$

where $\alpha_i = k_\perp^2 \rho_i^2/2$ and I_0 is the modified Bessel function. By quasineutrality, Eq. (54) also gives us the electron density perturbation:

$$\frac{\delta n_e}{n_{0e}} = -\frac{Z}{\tau} (1 - \hat{\Gamma}_0) \frac{e\varphi}{T_{0e}}. \quad (56)$$

We have recovered a rather popular standard treatment of the ion FLR.[86] Equation (56) is often referred to as the gyrokinetic Poisson equation. Note that Eq. (56) is consistent with the ordering for the density perturbation assumed in Sec. II B [see Eq. (17)].

It will be useful to remember that for $k_\perp \rho_i \ll 1$, $\Gamma_0(\alpha_i) \simeq 1 - \alpha_i$ and so

$$\frac{Z}{\tau} (1 - \Gamma_0) \simeq k_\perp^2 \rho_s^2 \quad (57)$$

in the long-wavelength limit—this leads to a significant simplification of Eq. (56). The long-wavelength form of the gyrokinetic Poisson equation is useful when one wishes to neglect the ion FLR effects. Formally, this can be done by means of the cold-ion approximation, $\tau \ll 1$. In view of Eq. (10), it immediately implies $k_\perp \rho_i \sim \sqrt{\tau} \ll 1$. Note that taking this limit does not require us to order τ within the β_e expansion as τ and β_e do not interfere with each other in

most of the ordering arguments of Sec. II B. The only exception to this is Eq. (24), which implies that if τ is excessively small, ion collisions (and, therefore, ion viscosity) might become important. The condition for this *not* to happen is $\beta_e \ll \tau^3$ or $\tau \gg (m_e/m_i)^{1/3}$. Clearly, it is safest to keep τ only moderately small and to treat any possible expansion in τ as subsidiary to the β_e expansion.

Finally, since δf_i given by Eq. (53) is even in $v_\parallel$ and gyroaveraging only involves $\mathbf{v}_\perp$, we have

$$u_{\parallel i} = \frac{1}{n_{0i}} \int d^3\mathbf{v} v_\parallel \langle \delta f_i \rangle_{\mathbf{r}} = 0. \tag{58}$$

This means that under the ordering, we have adopted the parallel current is carried predominantly by the electrons [see Eq. (37)].

Equations (56) and (58) together with Eqs. (39)–(42) constitute a complete set. We will assemble and summarize them momentarily, but first let us work out the collisional terms in our equations.

D. Collisions

In order to make our equations useable, we must calculate the collision terms. We would like to do this by employing a maximally simplified collision operator rather than the quantitatively correct but cumbersome Landau one. In order to choose the most appropriate model, we notice first that the electron kinetic equation does not involve any nontrivial evolution of the $v_\perp$ structure in the distribution function. We, therefore, believe it is acceptable to treat the kinetic equation as one-dimensional in velocity space—formally, this can be thought of as integrating out the $v_\perp$ dependence in g_e and F_{0e}. The simplest one-dimensional collision operator is the Lenard–Bernstein one,[87] which we write in the following form:

$$\left(\frac{\partial h_e}{\partial t}\right)_c = \nu_{ei}\left[\frac{1}{2}\frac{\partial}{\partial \hat{v}_\parallel}\left(\frac{\partial}{\partial \hat{v}_\parallel} + 2\hat{v}_\parallel\right) h_e + \frac{2\hat{v}_\parallel u_{\parallel i}}{v_{\mathrm{the}}} F_{0e} + \left(1 - 2\hat{v}_\parallel^2\right) F_{0e} \frac{1}{n_{0e}} \int dv'_\parallel \left(1 - 2\hat{v}'^2_\parallel\right) h_e\right], \tag{59}$$

where $\hat{v}_\parallel = v_\parallel / v_{\mathrm{the}}$ and the additional, non-differential terms in the operator have been constructed in such a way that electron momentum evolution is captured correctly, particle number and parallel kinetic energy are conserved and the H theorem is satisfied (cf. Ref. 88; the H theorem will be proved explicitly in Sec. V B). The collision frequency ν_{ei} is taken to be velocity-independent—a gross simplification compared to the uncensored reality, in which it is in fact a strong function of velocity. We do not believe this to be an important shortcoming of our model, however, because our focus will be on the role of the collision operator as small-scale regularization in velocity space rather than on quantitatively precise calculations of the effects of finite collisionality.

Let us now calculate the collisional terms in Eqs. (40) and (42). Using Eq. (33) for h_e and Eq. (37) for $u_{\parallel e}$, we get

$$\frac{1}{n_{0e}} \int d^3\mathbf{v} v_\parallel \left(\frac{\partial h_e}{\partial t}\right)_c = \nu_{ei}\left(u_{\parallel i} - u_{\parallel e}\right) = -\frac{e\nu_{ei}}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel, \tag{60}$$

which simply gives rise to a resistive diffusion term in Eq. (40) with Ohmic diffusivity $\eta = \nu_{ei} d_e^2$. The collision operator defined by Eq. (43) becomes

$$C[g_e] = \nu_{ei}\left[\frac{1}{2}\frac{\partial}{\partial \hat{v}_\parallel}\left(\frac{\partial}{\partial \hat{v}_\parallel} + 2\hat{v}_\parallel\right) g_e - \left(1 - 2\hat{v}_\parallel^2\right)\frac{\delta T_{\parallel e}}{T_{0e}} F_{0e}\right], \tag{61}$$

where we have used Eqs. (33) and (60), the properties of g_e [Eq. (36)] and the definition of the parallel electron temperature perturbation [Eq. (41)].

E. Summary of the equations

Assembling now Eqs. (39)–(42), (56), (58), and (60), we arrive at the following closed set

$$\frac{d}{dt}\frac{Z}{\tau}\left(1 - \hat{\Gamma}_0\right)\frac{e\varphi}{T_{0e}} = \hat{\mathbf{b}} \cdot \nabla \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel, \tag{62}$$

$$\frac{d}{dt}\left(A_\parallel - d_e^2 \nabla_\perp^2 A_\parallel\right) = \eta \nabla_\perp^2 A_\parallel - c\frac{\partial \varphi}{\partial z} - \frac{cT_{0e}}{e}\hat{\mathbf{b}} \cdot \nabla\left[\frac{Z}{\tau}\left(1 - \hat{\Gamma}_0\right)\frac{e\varphi}{T_{0e}} - \frac{\delta T_{\parallel e}}{T_{0e}}\right], \tag{63}$$

$$\frac{dg_e}{dt} + v_\parallel \hat{\mathbf{b}} \cdot \nabla\left(g_e - \frac{\delta T_{\parallel e}}{T_{0e}} F_{0e}\right) = C[g_e] + \left(1 - \frac{2v_\parallel^2}{v_{\mathrm{the}}^2}\right) F_{0e} \hat{\mathbf{b}} \cdot \nabla \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel, \tag{64}$$

where the following short-hand notation is used

$$\frac{\delta T_{\parallel e}}{T_{0e}} = \frac{1}{n_{0e}} \int d^3\mathbf{v} \frac{2v_\parallel^2}{v_{\mathrm{the}}^2} g_e, \tag{65}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{c}{B_0}\{\varphi, \ldots\}, \tag{66}$$

$$\hat{\mathbf{b}} \cdot \nabla = \frac{\partial}{\partial z} - \frac{1}{B_0}\{A_\parallel, \ldots\}, \tag{67}$$

$\eta = \nu_{ei} d_e^2$ is the Ohmic diffusivity and $C[g_e]$ is the collision operator—a simple model for which, based on the Lenard–Bernstein operator, is given by Eq. (61). These equations evolve three fields: φ, $A_\parallel$, and g_e, which are all functions of time and three spatial coordinates; g_e is also a function of $v_\parallel$ and $v_\perp$, although the $v_\perp$ dependence can be ignored (or integrated out) if the Lenard–Bernstein collision operator is used.

Equations (62)–(64) constitute a minimal physically realizable paradigm for magnetic reconnection and, more generally, low-frequency nonlinear plasma dynamics with a strong guide field, including all effects we expect to be important: ion sound scale physics, ion FLR, electron inertia, electron collisions, Ohmic resistivity, and electron

temperature perturbation determined by a kinetic equation. We will refer to these equations as *Kinetic Reduced Electron Heating Model (KREHM)*. Its hybrid fluid-kinetic nature and the presence of the kinetic electron heating channel (further discussed below), constitutes the main difference with previously considered models: in the 2D, collisionless case, our equations can be manipulated into a form similar to that proposed by de Blank;[53,54] by setting $g_e = 0$ (isothermal-electrons closure), we recover the equations of Schep *et al.*[37] While the use of a fluid model ($g_e = 0$) might be a useful simplification, we stress that setting $g_e = 0$ cannot be rigorously justified (at least in the analytical framework we have chosen): indeed, $g_e = 0$ is not a solution of Eq. (64) unless $\hat{\mathbf{b}} \cdot \nabla \nabla_\perp^2 A_\parallel = 0$ (i.e., $\hat{\mathbf{b}} \cdot \nabla j_\parallel = 0$), which cannot be the case in a reconnection-relevant solution.

We will show in Sec. IV that in the collisionless (or, more precisely, weakly collisional) limit, the coupling of the fluid system to the kinetic equation (64) via parallel electron temperature fluctuations provides the electron heating channel and so makes collisionless reconnection thermodynamically irreversible. We believe this to be fundamentally important and physically the most interesting outcome of our calculation. In the remainder of the paper we will concentrate on this aspect.

We stress that although the kinetic coupling leads to irreversibility and heating, it does *not* constitute a flux-unfreezing mechanism by itself; magnetic field lines can only be broken by resistivity and electron inertia (a simple proof of this statement was provided in Sec. III B).

To conclude this brief summary of our equations, let us note that a number of well-known limiting cases are easily derivable from them. As already discussed, in the collisionless limit, for $g_e = 0$, they reduce to the two-fluid model for strong-guide-field collisionless reconnection.[37] In the collisionally dominated limit, Eq. (64) reduces to the standard equation for the evolution of the parallel electron temperature via parallel heat conduction (see Sec. V C). If collisions are so large that the resistive scale is larger than both ρ_s and d_e, Eqs. (62) and (63) reduce straightforwardly to the standard Reduced MHD equations.[89] In the intermediate limit $k_\perp \rho_i \gg 1$, $k_\perp d_e \ll 1$, they reduce to the low-beta limit of the electron reduced MHD equations,[49] and so can support kinetic Alfvén waves. More generally, we show in Appendix B how the full collisionless gyrokinetic dispersion relation in the asymptotic limit of $k_\perp \rho_e \ll 1$ and low beta[61] is recovered from Eqs. (62)–(64). Finally, for the linear tearing mode, the paradigmatic linear problem of magnetic reconnection theory, Eqs. (62)–(64) recover the correct kinetic formulation both for the collisionless and the semicollisional regimes[67]—this topic is also treated in detail in Appendix B.

IV. ENERGETICS

A. Free energy

A broad class of δf kinetic systems, including gyrokinetics, conserve (in the absence of collisions) a positive-definite quadratic quantity that has the physical meaning of the free energy of the combined system of particles and perturbed fields and plays the role of the generalized energy invariant (see Refs. 48, 49, and references therein):

$$W = \sum_s \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0s} \delta f_s^2}{2F_{0s}} + \int \frac{d^3\mathbf{r}}{V} \frac{|\delta \mathbf{B}|^2}{8\pi} = -\sum_s T_{0s} \delta S_s + U, \tag{68}$$

where δS_s is the perturbed entropy of species s and U is the energy of the perturbed magnetic field. Since energy flows play a fundamental role in all nonlinear phenomena, it will be instructive to understand the energetics of our equations. As KREHM is a particular limit of gyrokinetics, this is done by specializing from the more general gyrokinetic case.

We saw in Sec. III A that

$$\delta f_e = \left(\frac{\delta n_e}{n_{0e}} + \frac{2 v_\parallel u_{\parallel e}}{v_{\mathrm{the}}^2} \right) F_{0e} + g_e \tag{69}$$

[see Eq. (33)]. Hence, we find immediately that the electron perturbed entropy is

$$-T_{0e} \delta S_e = \int \frac{d^3\mathbf{r}}{V} \left(\frac{n_{0e} T_{0e}}{2} \frac{\delta n_e^2}{n_{0e}^2} + \frac{m_e n_{0e} u_{\parallel e}^2}{2} + \int d^3\mathbf{v} \frac{T_{0e} g_e^2}{2F_{0e}} \right). \tag{70}$$

The three terms here are the electron density variance (denoted Y and, as we shall see, interpretable as generalized enstrophy), the kinetic energy of the parallel electron flow (denoted K_e) and the free energy associated with the reduced electron distribution function g_e, which we will refer to as *electron free energy* and denote H_e. Using Eqs. (37), (56), and (58), we rewrite the above expression as follows:

$$\begin{aligned} -T_{0e} \delta S_e &= Y + K_e + H_e \\ &= \frac{n_{0e} T_{0e}}{2} \sum_{\mathbf{k}} \frac{Z^2}{\tau^2} [1 - \Gamma_0(\alpha_i)]^2 \frac{e^2 |\varphi_{\mathbf{k}}|^2}{T_{0e}^2} \\ &\quad + \int \frac{d^3\mathbf{r}}{V} \frac{d_e^2 |\nabla_\perp^2 A_\parallel|^2}{8\pi} + \int \frac{d^3\mathbf{r}}{V} \int d^3\mathbf{v} \frac{T_{0e} g_e^2}{2F_{0e}}, \end{aligned} \tag{71}$$

where Γ_0 was defined in Eq. (55).

Using Eq. (53) for δf_i, we find the ion perturbed entropy:

$$-T_{0i} \delta S_i = \frac{n_{0e} T_{0e}}{2} \sum_{\mathbf{k}} \frac{Z}{\tau} [1 - \Gamma_0(\alpha_i)] \frac{e^2 |\varphi_{\mathbf{k}}|^2}{T_{0e}^2} \equiv K_i. \tag{72}$$

Note that, were this quantity restricted to $k_\perp \rho_i \ll 1$, it would simply be the kinetic energy of the $\mathbf{E} \times \mathbf{B}$ flows: indeed, using Eq. (57) and assuming an expansion in τ (as explained just after the latter formula), the quantity given by Eq. (72) becomes

$$K_i = \int \frac{d^3\mathbf{r}}{V} \frac{e^2 n_{0e}}{2T_{0e}} \rho_s^2 |\nabla_\perp \varphi|^2 = \int \frac{d^3\mathbf{r}}{V} \frac{m_i n_{0i} u_\perp^2}{2}, \tag{73}$$

where $u_\perp = c|\nabla_\perp \varphi|/B_0$ is the $\mathbf{E} \times \mathbf{B}$ flow velocity. Under the same approximation, the first term on the right-hand side

of Eq. (71) (which arose from the electron density variance) is recognizable as the enstrophy of the $\mathbf{E}\times\mathbf{B}$ flow:

$$Y=\int\frac{d^3\mathbf{r}}{V}\frac{e^2n_{0e}}{2T_{0e}}\rho_s^4|\nabla_\perp^2\varphi|^2. \tag{74}$$

Finally, the magnetic energy is

$$U=\int\frac{d^3\mathbf{r}}{V}\frac{\delta B_\perp^2}{8\pi}=\int\frac{d^3\mathbf{r}}{V}\frac{|\nabla_\perp A_\parallel|^2}{8\pi} \tag{75}$$

because $\delta B_\parallel$ is subdominant under our ordering [Eq. (21)]. Combining Eqs. (71), (72), and (75) to reassemble the total free energy [Eq. (68)], we get

$$\begin{aligned}W&=K_i+Y+U+K_e+H_e\\&=\sum_{\mathbf{k}}\left[1+\frac{Z}{\tau}(1-\Gamma_0)\right]\frac{Z}{\tau}(1-\Gamma_0)\frac{e^2n_{0e}|\varphi_{\mathbf{k}}|^2}{2T_{0e}}\\&\quad+\int\frac{d^3\mathbf{r}}{V}\frac{|\nabla_\perp A_\parallel|^2+d_e^2|\nabla_\perp^2A_\parallel|^2}{8\pi}\\&\quad+\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0e}g_e^2}{2F_{0e}}.\end{aligned} \tag{76}$$

In three dimensions (3D), W is the only quadratic invariant of our equations. In two dimensions (2D), there is an additional family of invariants (inherited from the more general 2D invariants of gyrokinetics[49])—they are worked out in Appendix A. Their existence, while opening interesting avenues of investigation of an academic kind, suggests that one should be very cautious in generalizing 2D analytical and numerical results to 3D reality (similarly to the situation in fluid turbulence theory, where nonlinear interactions and energy flows are dramatically different in 2D and in 3D).

B. Energy exchange between fields

It is illuminating to consider separately the time evolution of the five constituent parts of W. Multiplying Eq. (62) by $e\varphi/T_{0e}$ or by $(Z/\tau)(1-\hat{\Gamma}_0)e\varphi/T_{0e}$ and integrating over space, we obtain the evolution of the ion kinetic energy and enstrophy, respectively:

$$\frac{dK_i}{dt}=\frac{c}{4\pi}\int\frac{d^3\mathbf{r}}{V}\varphi\hat{\mathbf{b}}\cdot\nabla\nabla_\perp^2A_\parallel=-\int\frac{d^3\mathbf{r}}{V}E_\parallel^{\mathrm{st}}j_\parallel, \tag{77}$$

$$\frac{dY}{dt}=\int\frac{d^3\mathbf{r}}{V}j_\parallel\hat{\mathbf{b}}\cdot\nabla\left[\frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi\right], \tag{78}$$

where $E_\parallel^{\mathrm{st}}=-\hat{\mathbf{b}}\cdot\nabla\varphi$ is the electrostatic part of the parallel electric field and $j_\parallel=-(c/4\pi)\nabla_\perp^2A_\parallel$ is the parallel current. Multiplying Eq. (63) by $\nabla_\perp^2A_\parallel/4\pi$ and integrating over space, we obtain the evolution of the combined magnetic and electron kinetic energies:

$$\begin{aligned}\frac{d}{dt}(U+K_e)=\int\frac{d^3\mathbf{r}}{V}\Bigg\{&-\frac{4\pi}{c^2}\eta j_\parallel^2+E_\parallel^{\mathrm{st}}j_\parallel\\&-j_\parallel\hat{\mathbf{b}}\cdot\nabla\left[\frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi\right]+\frac{1}{e}j_\parallel\hat{\mathbf{b}}\cdot\nabla\delta T_{\parallel e}\Bigg\}.\end{aligned} \tag{79}$$

With the help of Eqs. (70)–(79), we now examine the evolution of the combined "fluid" (electromagnetic) part of the free energy, $W_{\mathrm{fluid}}=K_i+Y+U+K_e$, which is the quantity that is normally considered to be conserved in two-fluid models of Hamiltonian collisionless reconnection.[37] The energy exchange terms containing φ cancel and we find that W_{fluid} can only change due to two effects: the resistive Ohmic dissipation [the first term on the right-hand side of Eq. (79)], and an exchange with the electron free energy H_e controlled by the last term on the right-hand side of Eq. (79). Since $j_\parallel=-en_{0e}u_{\parallel e}$ [see Eqs. (37) and (58)], we can interpret this term as work done by the parallel electron pressure $\delta p_{\parallel e}=n_{0e}\delta T_{\parallel e}$:

$$\int\frac{d^3\mathbf{r}}{V}\frac{1}{e}j_\parallel\hat{\mathbf{b}}\cdot\nabla\delta T_{\parallel e}=-\int\frac{d^3\mathbf{r}}{V}u_{\parallel e}\hat{\mathbf{b}}\cdot\nabla\delta p_{\parallel e}\equiv-Q. \tag{80}$$

Thus, the fluid part of the free energy evolves according to

$$\frac{d}{dt}W_{\mathrm{fluid}}=-Q-\frac{4\pi}{c^2}\eta\int\frac{d^3\mathbf{r}}{V}j_\parallel^2. \tag{81}$$

Note that Q is not sign-definite and can correspond both to loss and gain of energy. In particular, it clearly represents a loss ($Q>0$) if the electron fluid is compressed.

The fluid energy lost or gained this way is recovered in the evolution of the electron free energy, which we obtain by multiplying Eq. (64) by $T_{0e}g_e/F_{0e}$ and integrating over the entire phase-space (positions and velocities):

$$\frac{dH_e}{dt}=Q-D_{\mathrm{coll}}, \tag{82}$$

where D_{coll} is the dissipation term due to collisions, which must be positive definite for any collision operator that satisfies Boltzmann's H theorem (see discussion in Ref. 88 and references therein):

$$D_{\mathrm{coll}}=-\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0e}g_eC[g_e]}{F_{0e}}\geq0. \tag{83}$$

Adding Eqs. (81) and (82), we find that the total free energy W is conserved in the absence of collisions, as stated at the beginning of this section.

C. Dissipation, electron heating and irreversibility of collisionless reconnection

Consider some initial configuration prone to reconnection. Such a configuration will possess a certain amount of magnetic energy U, which in the process of reconnection will be converted into some other form. There can be two fundamentally different types of such energy conversion.

First, since the conserved quantity is not U but the total free energy $W=U+K_e+K_i+Y+H_e$, the magnetic energy can be transferred dynamically into K_e, K_i, Y, or H_e without

loss of W, or increase of entropy—therefore, in principle, reversibly.

Second, the magnetic energy can be dissipated via the resistive term in Eq. (81) and/or via the collisional dissipation term $D_{\rm coll}$ in Eq. (82)—in the latter case, it first has to be converted into the electron free energy via the energy exchange term Q in Eqs. (81) and (82). These processes are irreversible because they involve conversion of the fluctuation energy into the thermal energy of the bulk electron distribution, or electron heating. Indeed, it is not hard to show[48,61] that

$$\frac{3}{2}n_{0e}\frac{dT_{0e}}{dt} = -\overline{\int\frac{d^3\mathbf{r}}{V}\int d^3\mathbf{v}\frac{T_{0e}\delta f_e}{F_{0e}}\left(\frac{\partial\delta f_e}{\partial t}\right)_c} = \overline{D_{\rm coll}} + \frac{4\pi}{c^2}\eta\int\frac{d^3\mathbf{r}}{V}\overline{j_\parallel^2}, \tag{84}$$

where Eq. (69) was used to express δf_e and overbar means averaging over dynamical timescales (in gyrokinetics, the rate of change of the mean equilibrium quantities is $\sim\epsilon^2\omega$, so the transport equations are obtained via intermediate time averaging[61,90]).

If we formally forbade any collisions at all and set $\nu_{ei}=0$ exactly, the exchanges between different constituent parts of W would be the only possible energy conversion mechanism. Indeed, Hamiltonian theories of collisionless reconnection based on the conservative fluid models ($g_e=0$, $W_{\rm fluid}=$ const) find that the nonlinear stage of a reconnecting mode corresponds to a transfer of magnetic energy U into ordered parallel electron kinetic energy K_e, perpendicular ion kinetic energy K_i, and enstrophy of the $\mathbf{E}\times\mathbf{B}$ flow Y. In this situation, the only way energy can be lost is via ejection of material—although the reconnection process remains technically reversible.

Is this, however, a good approximation of what will in fact occur? We believe that *a priori* it is not because, even if the collisionality of the plasma is small, the dissipation terms cannot be neglected as large spatial gradients will imply finite currents and large velocity-space gradients will imply finite values of $C[g_e]$. The latter effect is especially important. Indeed, if resistivity (Joule heating) is ignored, magnetic energy can still be converted into electron heat via transfer into H_e, formation of small-scale structure in $v_\parallel$ by means of linear phase-mixing (the second term on the left-hand side in Eq. (64); see further discussion in Sec. V) and the consequent collisional dissipation of the resulting fine-scale electron distribution: since the collision operator is a diffusion operator in $v_\parallel$ [see Eq. (61)], $D_{\rm coll}$ will have a finite value provided structure develops in the velocity-space that satisfies

$$\nu_{ei}\frac{\partial^2}{\partial\hat{v}_\parallel^2}\gtrsim\omega \quad\Rightarrow\quad \frac{\delta v_\parallel}{v_{\rm the}}\lesssim\left(\frac{\nu_{ei}}{\omega}\right)^{1/2}. \tag{85}$$

This phase-mixing channel of electron heating is open even in very weakly collisional plasmas. It seems *a priori* clear that its presence should be qualitatively important as it breaks the Hamiltonian nature of the problem and renders the process of "collisionless" reconnection irreversible.

The question of course remains whether the phase mixing will be important in any given physical situation of interest. Since most such situations are nonlinear, it is difficult to provide a universal answer to this question beyond the general argument that nature rarely ignores an energy dissipation channel if one is available. Here it is opportunity to remind the reader that there have been several recent numerical studies of nonlinear kinetic reconnection that reported the non-negligibility of the parallel electron temperature gradient term in the generalized Ohm's law[35,45,46,52] [see Eq. (63)]—and, therefore, the non-negligibility of the energy exchange term Q in Eq. (81), which ultimately transfers free energy into electron heat [Eq. (82)]. It is a testable prediction that in all such cases, small-scale structure should be generated in the velocity space (see Sec. V D), so as to enable the electron heating channel.

In Sec. V, we discuss the electron kinetics a little further, in particular quantifying the notion of the free-energy transfer to small parallel scales in velocity space. We will conclude (in Sec. V E) that for configurations that evolve sufficiently slowly in time, the electron heating rate is finite and independent of the collision frequency as $\nu_{ei}\to+0$.

V. VELOCITY-SPACE DYNAMICS

A. Hermite expansion

The velocity-space dynamics and the emergence of small-scale structure in $v_\parallel$ are best understood in terms of the expansion of the electron distribution function g_e in Hermite polynomials.[91–95] Let

$$g_e(v_\parallel) = \sum_{m=0}^{\infty}\frac{H_m(\hat{v}_\parallel)}{\sqrt{2^m m!}}\hat{g}_m F_{0e}(v_\parallel), \tag{86}$$

where $\hat{v}_\parallel = v_\parallel/v_{\rm the}$, perpendicular velocity dependence is understood to have been integrated out, the Hermite polynomials are

$$H_m(\hat{v}_\parallel) = (-1)^m e^{\hat{v}_\parallel^2}\frac{d^m}{d\hat{v}_\parallel^m}e^{-\hat{v}_\parallel^2} \tag{87}$$

(so $H_0=1$, $H_1=2\hat{v}_\parallel$, $H_2=4\hat{v}_\parallel^2-2$, etc.) and $\hat{g}_m$ are the Hermite expansion coefficients (dimensionless), which can be calculated according to

$$\hat{g}_m = \frac{1}{n_{0e}}\int_{-\infty}^{+\infty}dv_\parallel\frac{H_m(\hat{v}_\parallel)}{\sqrt{2^m m!}}g_e(v_\parallel). \tag{88}$$

In view of Eq. (36), we must have $\hat{g}_0=\hat{g}_1=0$. Equation (41) implies, by definition, that

$$\hat{g}_2 = \frac{1}{\sqrt{2}}\frac{\delta T_{\parallel e}}{T_{0e}}. \tag{89}$$

Using the orthogonality of Hermite polynomials,

$$\frac{1}{n_{0e}}\int_{-\infty}^{+\infty}dv_\parallel\frac{H_m(\hat{v}_\parallel)H_n(\hat{v}_\parallel)}{2^m m!}F_{0e} = \delta_{mn}, \tag{90}$$

we note that the electron free energy in terms of the Hermite coefficients is

$$H_e = \int \frac{d^3\mathbf{r}}{V} \int_{-\infty}^{+\infty} dv_\parallel \frac{T_{0e} g_e^2}{2F_{0e}} = \int \frac{d^3\mathbf{r}}{V} \frac{n_{0e} T_{0e}}{2} \sum_{m=2}^{\infty} \hat{g}_m^2. \quad (91)$$

Taking the Hermite transform [Eq. (88)] of Eq. (64) and using the recursive property of the Hermite polynomials,

$$\hat{v}_\parallel H_m(\hat{v}_\parallel) = \frac{1}{2} H_{m+1}(\hat{v}_\parallel) + m H_{m-1}(\hat{v}_\parallel), \quad (92)$$

we arrive at

$$\frac{d\hat{g}_m}{dt} + v_{\mathrm{the}} \hat{\mathbf{b}} \cdot \nabla \left(\sqrt{\frac{m+1}{2}} \hat{g}_{m+1} + \sqrt{\frac{m}{2}} \hat{g}_{m-1} - \delta_{m,1} \hat{g}_2 \right) = -\sqrt{2} \delta_{m,2} \hat{\mathbf{b}} \cdot \nabla \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel - \nu_{ei} \left(m \hat{g}_m - 2\delta_{m,2} \hat{g}_2 \right). \quad (93)$$

Note that Hermite polynomials are eigenfunctions of the Lenard–Bernstein operator (61).

It is not hard to see that Eq. (93) is consistent with $\hat{g}_0 = \hat{g}_1 = 0$. We would like to recast the equation for $\hat{g}_2$ in terms of the electron temperature perturbation with the aid of Eq. (89):

$$\frac{d}{dt} \frac{\delta T_{\parallel e}}{T_{0e}} + v_{\mathrm{the}} \hat{\mathbf{b}} \cdot \nabla \sqrt{3}\, \hat{g}_3 = -2\hat{\mathbf{b}} \cdot \nabla \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel. \quad (94)$$

This equation is coupled to the rest of the kinetics via the heat flux proportional to $\hat{g}_3$ (the second term on the left-hand side). For $m \geq 3$, we have

$$\frac{d\hat{g}_m}{dt} + v_{\mathrm{the}} \hat{\mathbf{b}} \cdot \nabla \left(\sqrt{\frac{m+1}{2}} \hat{g}_{m+1} + \sqrt{\frac{m}{2}} \hat{g}_{m-1} \right) = -\nu_{ei} m \hat{g}_m. \quad (95)$$

Equation (95) shows that Hermite modes of order m are coupled both to lower $(m-1)$ and higher $(m+1)$ modes. Collisions provide a cutoff at sufficiently large m regardless of the magnitude of the collision frequency. We will discuss this further in Sec. V D.

Together with Eqs. (62) and (63), Eqs. (94) and (95) can be solved as a system of partial differential equations in the three- or two-dimensional position space. This appears to be an attractive way to carry out numerical simulations of collisionless (in fact, weakly collisional) reconnection or, indeed, of other kinetic phenomena in strongly magnetized plasmas. This system is substantially simpler than the full kinetic[31–35] or gyrokinetic[42–46] descriptions that have been used to study reconnection or plasma turbulence[52,96] so far (see Sec. V D for estimates of how many Hermite modes must be kept for any given collisionality). The first numerical study using our equations is reported in Ref. 97.

The Hermite formalism allows us to provide very concise derivations of three important results: the H theorem for our collision operator (Sec. V B), the so-called semicollisional limit of our equations, in which collisionality dominates and the parallel electron temperature is determined by a fluid equation (Sec. V C), the Hermite "spectrum," which quantifies the fine structure in the velocity space caused by the parallel phase mixing (Sec. V D), and finally the electron heating rate (Sec. V E).

B. *H* theorem

As we mentioned in Sec. III D, a key requirement for choosing a model collision operator is that it satisfies Boltzmann's H theorem, i.e., that collisional dissipation leads to increase of entropy. This means that we must have $D_{\mathrm{coll}} \geq 0$ [see Eq. (83)]. Substituting Eq. (86), using the collision operator given by Eq. (61) and the orthogonality of Hermite polynomials [Eq. (90)], we get

$$\frac{1}{n_{0e}} \int_{-\infty}^{+\infty} dv_\parallel \frac{g_e C[g_e]}{F_{0e}} = -\nu_{ei} \sum_{m=0}^{\infty} m \hat{g}_m^2 + 2\nu_{ei} \hat{g}_2 = -\nu_{ei} \sum_{m=3}^{\infty} m \hat{g}_m^2 \leq 0, \quad (96)$$

so $D_{\mathrm{coll}} \geq 0$, q.e.d.

C. Semicollisional limit

Consider the semicollisional limit, $\nu_{ei} \gg \omega$ and $k_\parallel \lambda_{\mathrm{mfpe}} \ll 1$ ["semicollisional" because the perpendicular microscale effects associated with ρ_s and ρ_i are retained, although the electron inertia term in Eq. (63) must be neglected compared with the resistive term]. From Eq. (95), it is clear that in this limit, the Hermite coefficients get smaller with m:

$$\frac{\hat{g}_m}{\hat{g}_{m-1}} \sim \frac{k_\parallel v_{\mathrm{the}}}{\sqrt{m}\nu_{ei}} = \frac{k_\parallel \lambda_{\mathrm{mfpe}}}{\sqrt{m}} \ll 1. \quad (97)$$

Therefore, Eq. (95) allows us to express higher-order coefficients in terms of the gradients of the lower-order ones and, in particular, $\hat{g}_3$ in terms of $\hat{g}_2$, i.e., the heat flux in term the temperature gradient:

$$\hat{g}_3 \simeq -\frac{1}{2\sqrt{3}} \frac{v_{\mathrm{the}}}{\nu_{ei}} \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_{\parallel e}}{T_{0e}}. \quad (98)$$

Substituting this into Eq. (94), we obtain the standard equation for the electron temperature:

$$\frac{d}{dt} \frac{\delta T_{\parallel e}}{T_{0e}} = \kappa_{\parallel e} \hat{\mathbf{b}} \cdot \nabla \left(\hat{\mathbf{b}} \cdot \nabla \frac{\delta T_{\parallel e}}{T_{0e}} \right) - 2\hat{\mathbf{b}} \cdot \nabla \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel, \quad (99)$$

where $\kappa_{\parallel e} = v_{\mathrm{the}}^2 / 2\nu_{ei}$ is the parallel (Spitzer) thermal diffusivity. In the semicollisional limit, Eq. (99) replaces Eq. (64) and completes what has become a purely fluid system.

The energetics of this fluid system are a simple particular case of the general situation discussed in Sec. IV. The electron free energy is [see Eq. (91)]

$$H_e = \int \frac{d^3\mathbf{r}}{V} \frac{n_{0e} T_{0e}}{2} \hat{g}_2^2 = \int \frac{d^3\mathbf{r}}{V} \frac{n_{0e} T_{0e}}{4} \frac{\delta T_{\parallel e}^2}{T_{0e}^2}. \quad (100)$$

From Eq. (99), we get an evolution equation for H_e equivalent to the general Eq. (82):

$$\frac{dH_e}{dt} = Q - \kappa_{\|e}\int \frac{d^3\mathbf{r}}{V}\frac{n_{0e}T_{0e}}{2}\left(\hat{\mathbf{b}}\cdot\nabla\frac{\delta T_{\|e}}{T_{0e}}\right)^2, \tag{101}$$

where Q is the energy exchange given by Eq. (80) and the dissipation term is the same as $D_{\rm coll}$ defined by Eq. (83)—this is checked by using Eq. (96) and restricting the sum just to $\hat{g}_3$, as given by Eq. (98).

D. Hermite spectrum

We now return to Eq. (95), linearize it and Fourier transform in the parallel direction. If we denote $\tilde{g}_m(k_\|) = (i\,\mathrm{sgn}k_\|)^m\hat{g}_m(k_\|)$, we can write the resulting equation in the following form:

$$\frac{\partial\tilde{g}_m}{\partial t} + |k_\||v_{\rm the}\left(\sqrt{\frac{m+1}{2}}\tilde{g}_{m+1} - \sqrt{\frac{m}{2}}\tilde{g}_{m-1}\right) = -\nu_{ei}m\tilde{g}_m, \tag{102}$$

which has the convenient property of supporting real solutions.[109] If we now define the Hermite spectrum as $E_m = |\hat{g}_m|^2/2 = \tilde{g}_m^2/2$, we find that it evolves according to

$$\frac{\partial E_m}{\partial t} = -\left(\Gamma_{m+1/2} - \Gamma_{m-1/2}\right) - 2\nu_{ei}mE_m, \tag{103}$$

where $\Gamma_{m-1/2} = |k_\||v_{\rm the}\sqrt{m/2}\tilde{g}_m\tilde{g}_{m-1}$ can be thought of as the flux of the electron free energy through the Hermite space. When $m \gg 1$, we may approximate $\Gamma_m \approx |k_\||v_{\rm the}\sqrt{2m}E_m$ and

$$\frac{\partial E_m}{\partial t} = -|k_\||v_{\rm the}\frac{\partial}{\partial m}\sqrt{2m}E_m - 2\nu_{ei}mE_m. \tag{104}$$

1. Hermite spectrum in steady state

Equation (104) has a steady-state solution[110]

$$E_m = \frac{C(k_\|)}{\sqrt{m}}\exp\left[-\left(\frac{m}{m_c}\right)^{3/2}\right], \tag{105}$$

where $C(k_\|)$ is some function of $k_\|$ (determined by the dynamics of φ, $A_\|$, and $\delta T_{\|e}$) and the collisional cutoff is

$$m_c = \left(\frac{3}{2\sqrt{2}}\frac{|k_\||v_{\rm the}}{\nu_{ei}}\right)^{2/3}. \tag{106}$$

When $m \ll m_c$, we have the spectrum scaling as $E_m \sim m^{-1/2}$ in Hermite space. This scaling implies that the electron free energy $H_e \propto \sum_{m=2}^{\infty} E_m$ is dominated by the Hermite modes $m \sim m_c$, which are transferring their energy into electron heat via collisions (see Sec. V E).

2. Hermite spectrum for growing modes

Consider now a slightly more general situation, in which the entire Hermite spectrum is growing at some rate 2γ (this would be the case, for example, for the tearing mode; see Appendix B 3). Then $\partial E_m/\partial t = 2\gamma E_m$ and the solution of Eq. (104) is

$$E_m = \frac{C(k_\|)}{\sqrt{m}}\exp\left[-\left(\frac{m}{m_\gamma}\right)^{1/2} - \left(\frac{m}{m_c}\right)^{3/2}\right], \tag{107}$$

where another cutoff has appeared, associated with the growth rate γ:

$$m_\gamma = \left(\frac{|k_\||v_{\rm the}}{2\sqrt{2}\gamma}\right)^2. \tag{108}$$

This cutoff supercedes the collisional cutoff (106) if $m_\gamma \ll m_c$, i.e., if the mode is growing so fast that

$$\gamma \gg (|k_\||v_{\rm the})^{2/3}\nu_{ei}^{1/3}. \tag{109}$$

Otherwise the steady-state spectrum (105) is recovered.

E. Electron heating rate

The results of Sec. V D allow us to make an estimate of the electron collisionless heating rate. Using Eqs. (83) and (96), we may write

$$D_{\rm coll} = -n_{0e}T_{0e}\sum_{k_\|}2\nu_{ei}\sum_{m=3}mE_m. \tag{110}$$

Approximating $\sum_{m=3} mE_m \approx \int_0^\infty dm\, mE_m$ and using Eq. (105), we find for the steady-state or slowly evolving modes,

$$D_{\rm coll} \approx -n_{0e}T_{0e}\sum_{k_\|}\sqrt{2}|k_\||v_{\rm the}C(k_\|) \tag{111}$$

independently of the collision frequency ν_{ei} as long as $m_c \gg 1$, i.e., $\nu_{ei} \ll |k_\||v_{\rm the}$. Thus, the electron heating rate is finite in the limit $\nu_{ei} \to +0$. This is analogous, e.g., to the situation in standard hydrodynamic turbulence where the rate of viscous dissipation has a finite limit at small viscosities because larger spatial gradients emerge[98]—or in sub-Larmor gyrokinetic turbulence, where a similar process involves large gradients both in space and in $v_\perp$.[48,49,59] The difference is that here the mechanism for the generation of small scales in $v_\|$ is linear (parallel particle streaming) rather than nonlinear (turbulent cascade).

In contrast, for modes growing fast enough to satisfy Eq. (109), a similar calculation using Eq. (107) gives

$$D_{\rm coll} \approx -n_{0e}T_{0e}\sum_{k_\|}\frac{\nu_{ei}}{2\sqrt{2}}\left(\frac{|k_\||v_{\rm the}}{\gamma}\right)^3 C(k_\|). \tag{112}$$

In the limit $\nu_{ei} \to +0$, $D_{\rm coll} \to 0$. Thus, linearly growing modes do not produce any electron heating in the collisionless limit, provided the growth rate continues to satisfy Eq. (109) in this limit.

VI. DISCUSSION

In the above, we have already provided the motivation behind this work as well as a non-mathematical preview (Sec. I),

a summary of our equations (Sec. III E) and the general arguments leading to the conclusion that electron heating must be an important dissipation channel (Sec. IV C). We developed this last topic somewhat further in Sec. V by setting forth what we believe to be a rather useful practical prescription for further numerical investigations, a conceptual view of the phase-space dynamics as a cascade in Hermite space, and a simple argument implying that the electron heating rate remains finite in the limit of vanishing collisionality (except for fast-growing linear modes).

We do not of course claim that our equations constitute a general theoretical framework for all types of magnetic reconnection. The assumption of low beta restricts their applicability only to very strongly magnetized plasmas (in Sec. II B 1, we indicated that typical real-world plasmas where beta is suitably low include the solar corona,[16] the LArge Plasma Device at UCLA[81] and edge regions in some tokamaks[82]). We do, however, believe that at least qualitatively, our equations might go beyond their formal domain of validity and capture much of the essential physics of gyrokinetic reconnection. We do not know how to derive a similarly simple and yet physically realizable model for high-beta and low-guide-field situations, which occur frequently in astrophysical contexts.

An unresolved limitation of our model concerns the problem of ultrasmall perpendicular scales. It has been found in several (2D) theoretical and numerical studies of various two-fluid models of collisionless reconnection that sub-d_e structures can form in the reconnection region, giving rise to unbounded gradients.[58,80,99,100] The only known way of taming these singularities is via hyperdiffusive terms in the generalized Ohm's law, provided, e.g., by electron viscosity. Since we do not have these terms in our equations, it is not guaranteed that the singularities can be dissipated via the electron heating channel.[111] If they are present, any practical numerical study of our equations will require additional hyperdiffusive regularization—a somewhat embarassing solution, since we would ideally like to be able to attribute all field-line breaking and electron heating to physical mechanisms that are legitimate under our ordering assumptions. We should like to note, however, that the evidence of singularities stems from 2D analyses of Hamiltonian reconnection models. This means that the phase space of the system is constrained both by the Hamiltonian structure and by the presence of multitudinous additional 2D invariants (see Appendix A), so it is not, in fact, inevitable that these singularities will still present in a 3D kinetic situation.

It seems clear that the next logical step is a numerical investigation of the solutions of our model. The key physical questions are whether reconnection is indeed made irreversible by the electron heating channel, how much energy is converted into heat, whether the reconnection is fast and what the structure of the reconnecting region is (X point? stable current sheet? unstable current sheet?). It would also be important to find out whether the equations are well posed without additional perpendicular regularization, i.e., if singularities form. All of these issues are likely to be tied up with the question of whether the 3D case behaves differently from 2D and is, therefore, the right case to investigate. Finally, from a technical point of view, our model requires fairly modest computational resources, so it may prove to be a nimble tool for scouting out the parameter space in preparation for the more computationally demanding gyrokinetic reconnection studies.[46,97]

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APPENDIX A: TWO-DIMENSIONAL INVARIANTS

1. Lagrangian conservation properties in 2D

Whereas the free energy [Eq. (68)] is the only known invariant of gyrokinetics in three dimensions (3D), formally restricting consideration to two dimensions (2D) introduces a host of new conservation properties. Indeed, consider the gyrokinetic equation Eq. (4) and with $\partial/\partial z = 0$ and no collisions. It can then be written as an advection equation of a single scalar quantity:

$$\frac{\partial \tilde{h}_s}{\partial t} + \frac{c}{B_0}\{\langle \chi \rangle_{\mathbf{R}_s}, \tilde{h}_s\} = 0, \tag{A1}$$

$$\tilde{h}_s = h_s - \frac{q_s \langle \chi \rangle_{\mathbf{R}_s}}{T_{0s}} F_{0s}. \tag{A2}$$

This means that the volume integral of any function of $\tilde{h}_s$ is conserved:

$$\frac{\partial}{\partial t}\int \frac{d^3\mathbf{R}_s}{V} f(\tilde{h}_s) = 0. \tag{A3}$$

Note that $\tilde{h}_s$ is a function of $v_\parallel$ and $v_\perp$, so the above equation defines an infinite set of invariants both in the sense of the arbitrariness of the function f and in the sense that the invariants are parametrized by the velocity variables.

Recalling our solution for the ion distribution function (Sec. III C), we find immediately that $\tilde{h}_i = g_i = 0$, the conservation law Eq. (A3) is satisfied trivially. For the electrons, using Eqs. (29), (33), (38), (56), and (58), we find

$$\tilde{h}_e = -\frac{e}{cm_e}\left(A_\parallel - d_e^2 \nabla_\perp^2 A_\parallel\right)\frac{2v_\parallel}{v_{\mathrm{the}}^2} F_{0e} - \frac{Z}{\tau}(1-\hat{\Gamma}_0)\frac{e\varphi}{T_{0e}} F_{0e} + g_e. \tag{A4}$$

This quantity obeys the advection equation (A1), which becomes

$$\frac{\partial \tilde{h}_e}{\partial t} + \frac{c}{B_0}\{\chi, \tilde{h}_e\} = 0, \tag{A5}$$

where $\chi = \varphi - (v_\parallel/c)A_\parallel$ [see Eq. (29)]. Thus, $\tilde{h}_e$, parametrized by $v_\parallel$, is an infinite family of "Lagrangian invariants," each with its own stream function $(c/B_0)\chi(v_\parallel)$. This is the "foliation" of the electron distribution function discussed in Refs. 55 and 56.

While (as argued in Refs. 55 and 56) this may be reminiscent of the Lagrangian properties of the two-fluid model,[37] setting $g_e = 0$ in Eq. (A4) does not return the Lagrangian invariants enjoyed by that model—those have to be derived separately and are particular to the isothermal-electrons closure (which is not a surprise because $g_e = 0$ is not a solution of our equations). For the record, in our notation, they are[37]

$$A_\pm = A_\parallel - d_e^2\nabla_\perp^2 A_\parallel \pm \frac{c\sqrt{2}}{v_{\mathrm{the}}}\frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi, \tag{A6}$$

advected by the effective potentials

$$\varphi_\mp = \varphi \mp \frac{v_{\mathrm{the}}}{c\sqrt{2}}A_\parallel \tag{A7}$$

[this is equivalent to picking $v_\parallel = \pm v_{\mathrm{the}}/\sqrt{2}$ and $g_e = 0$ in Eqs. (A4) and (A5)]. Note that $c\sqrt{2}/v_{\mathrm{the}} = cd_e/v_A\rho_s$. It is perhaps useful to give the evolution equations for $A_\pm$ with three–dimensionality, Ohmic resistivity, and nonisothermal electrons retained, to show how the Lagrangian invariants are broken by all of these effects:

$$\left(\frac{\partial}{\partial t} \pm \frac{v_{\mathrm{the}}}{\sqrt{2}}\frac{\partial}{\partial z}\right)A_\pm + \frac{c}{B_0}\{\varphi_\mp, A_\pm\} = -c\frac{\partial \varphi_\mp}{\partial z} + \eta\nabla_\perp^2 A_\parallel + \hat{\mathbf{b}}\cdot\nabla\frac{c}{e}\delta T_{\parallel e}. \tag{A8}$$

For a model with no collisions, isothermal electrons and cold ions, these equations reduce to those of Ref. 101. In the latter reference, its authors point out that, while the Lagrangian invariance of $A_\pm$ is lost in their 3D system, there is a cross-helicity invariant that continues to be conserved. We see, however, that this conservation law does not survive the inclusion of nonisothermal electrons:

$$\frac{d}{dt}\int\frac{d^3\mathbf{r}}{V}\frac{A_+^2 - A_-^2}{2} = \frac{2\sqrt{2}c}{v_{\mathrm{the}}}\int\frac{d^3\mathbf{r}}{V}\left(\eta\nabla_\perp^2 A_\parallel + \hat{\mathbf{b}}\cdot\nabla\frac{c}{e}\delta T_{\parallel e}\right)\frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi, \tag{A9}$$

another constraint on the evolution of the system that disappears in our treatment.

2. Quadratic 2D invariants

It is usually the quadratic invariants that are of most practical interest in nonlinear dynamics (or at least they are the ones that have the most physically transparent consequences, e.g., the inverse cascades in 2D gyrokinetic turbulence[102]). In gyrokinetics, specializing to the quadratic function f in Eq. (A3), gives one the following family of 2D gyrokinetic invariants:[49]

$$I_s = \int d^3\mathbf{v}\int\frac{d^3\mathbf{R}_s}{V}\frac{T_{0s}}{2F_{0s}}\left(h_s - \frac{q_s\langle\chi\rangle_{\mathbf{R}_s}}{T_{0s}}F_{0s}\right)^2. \tag{A10}$$

We have also now integrated over the velocity space to produce a conservation law that globally constrains the system in the phase space. Using Eq. (A4) to work out the electron invariant, we get

$$I_e = Y + \frac{1}{d_e^2}X + 2U + K_e + H_e = W + \frac{1}{d_e^2}X + U - K_i, \tag{A11}$$

where the definitions of, and evolution equations for the total free energy W, enstrophy Y, magnetic energy U, electron kinetic energy K_e, electron free energy H_e, and ion kinetic energy K_i can be found in Sec. IV A and the quantity

$$X = \int\frac{d^3\mathbf{r}}{V}\frac{A_\parallel^2}{8\pi}, \tag{A12}$$

does not have a commonly agreed name, but is sometimes referred to as "$A_\parallel^2$-stuff". It is conserved in standard 2D magnetohydrodynamics (MHD). The 2D conservation law we are about to derive is a generalization of this MHD result.

Equation (A11) tells us that the electron 2D invariant I_e contains the total free energy W, which is itself conserved both in 2D and in 3D. The remainder of I_e is, therefore, a 2D invariant on its own. After straightforward algebra analogous to the calculations in Sec. IV B, we find the evolution equation for this quantity:

$$\frac{d}{dt}\left[X + d_e^2(U - K_i)\right] = -\eta U + \frac{c}{4\pi}\int\frac{d^3\mathbf{r}}{V}\frac{\partial A_\parallel}{\partial z}\left\{\left[1 + \frac{Z}{\tau}(1-\hat{\Gamma}_0)\right]\varphi - d_e^2\nabla_\perp^2\varphi - \frac{1}{e}\delta T_{\parallel e}\right\}. \tag{A13}$$

This is a 2D collisionless invariant because the last term on the right-hand side vanishes if (and, generally speaking, only if) $\partial/\partial z = 0$.

We will not delve any further into the mathematical consequences of the 2D conservation laws and instead limit ourselves to remarking that the presence of so many constraints on the dynamics particular to the exactly 2D case ought to make one beware of too much optimism about the relevance of 2D results and intuitions to 3D dynamics.

APPENDIX B: LINEAR THEORY

1. Linearized equations

Since we will want to use the linearized equations that are about to be derived for treating the tearing mode problem, we would like to do the linearization around an equilibrium containing both the guide field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and some in-plane field—gyrokinetically speaking, this in-plane field is part of the small perturbation of the guide field. Thus, we let

$$\mathbf{B} = B_0\hat{\mathbf{z}} + \delta B_y^{(0)}(x)\hat{\mathbf{y}} + \delta\mathbf{B}_\perp^{(1)}, \tag{B1}$$

where $\delta B_\perp^{(1)} \ll \delta B_y^{(0)} \ll B_0$. Therefore, $A_\parallel = A_\parallel^{(0)}(x) + A_\parallel^{(1)}$, where $\delta B_y^{(0)} = -dA_\parallel^{(0)}/dx \equiv B_0 f(x)$ and $\delta\mathbf{B}_\perp^{(1)} = -\hat{\mathbf{z}}\times\nabla A_\parallel^{(1)}$. The function $f(x)$ will contain all the information about the in-plane equilibrium. Examining Eqs. (62)–(64), it is not hard to see that an equilibrium of this form can be maintained by letting $\varphi^{(0)} = 0$, $g_e^{(0)} = 0$ and adding an equilibrium parallel electric field to Eq. (63) to maintain the equilibrium magnetic field against the Ohmic resistivity: $cE_\parallel^{(0)} = \eta B_0 f'(x)$.

Inserting all this into Eqs. (62)–(64), dropping the superscripts on the perturbed quantities and Fourier transforming with respect to y, z and time, we get

$$-\omega\frac{Z}{\tau}(1-\hat{\Gamma}_0)\frac{e\varphi}{T_{0e}} = \frac{e}{cm_e}d_e^2[k_\parallel(x)\nabla_\perp^2 A_\parallel - k_y f''(x)A_\parallel], \tag{B2}$$

$$-\omega\left[A_\parallel - d_e^2\nabla_\perp^2 A_\parallel\right] = -i\eta\nabla_\perp^2 A_\parallel + k_y c d_e^2 f''(x)\varphi - k_\parallel(x)c\left\{\left[1+\frac{Z}{\tau}(1-\hat{\Gamma}_0)\right]\varphi - \frac{1}{e}\delta T_{\parallel e}\right\}, \tag{B3}$$

$$-\omega g_e + k_\parallel(x)v_\parallel\left(g_e - \frac{\delta T_{\parallel e}}{T_{0e}}F_{0e}\right) = -iC[g_e] + \left(1-\frac{2v_\parallel^2}{v_{\mathrm{the}}^2}\right)F_{0e}\frac{e}{cm_e}d_e^2[k_\parallel(x)\nabla_\perp^2 A_\parallel - k_y f''(x)A_\parallel], \tag{B4}$$

where $k_\parallel(x) = k_z + f(x)k_y$ and $\nabla_\perp^2 = \partial_x^2 - k_y^2$.

a. Collisionless limit

The simplest approach to linear theory in a kinetic plasma is to consider the purely collisionless case. Although, as we explained above, this is not really a good limit, even in a weakly collisional plasma, the linear results obtained in this limit are useful in that they will allow us to make contact with the existing theories. The collisionless limit is also a useful route to certain linear results (like Landau damping) that do in fact depend on infinitesimal amount of velocity-space dissipation.

Thus, let us set $\eta = 0$ and $C[g_e] = 0$ in Eqs. (B3) and (B4), respectively. If we now solve Eq. (B4) for g_e explicitly and then integrate over velocity space according to Eq. (65) to obtain the parallel electron temperature perturbation, we get, after standard algebra,

$$\frac{\delta T_{\parallel e}}{T_{0e}} = \frac{2}{|k_\parallel(x)|v_{\mathrm{the}}}\frac{Z(\zeta(x)) + \zeta(x)Z'(\zeta(x))}{Z'(\zeta(x))} \times \frac{e}{cm_e}d_e^2\left[k_\parallel(x)\nabla_\perp^2 A_\parallel - k_y f''(x)A_\parallel\right], \tag{B5}$$

where $\zeta(x) = \omega/|k_\parallel(x)|v_{\mathrm{the}}$, $Z(\zeta)$ is the plasma dispersion function[103] (not to be confused with Z in the ion charge Ze) and $Z'(\zeta) = -2[1+\zeta Z(\zeta)]$. All kinetic effects are wrapped up in the above expression for the parallel electron temperature perturbation. Using Eq. (B2), we can simplify it somewhat:

$$\frac{1}{e}\delta T_{\parallel e} = -2\zeta\frac{Z(\zeta)+\zeta Z'(\zeta)}{Z'(\zeta)}\frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi = \left[1-2\zeta^2+\frac{2}{Z'(\zeta)}\right]\frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi. \tag{B6}$$

This can now be substituted into Eq. (B3), whereupon Eqs. (B2) and (B3) form a closed set.

Recalling the gyrokinetic Poisson equation (56), Eq. (B6) can be interpreted as an equation of state with ζ-dependent effective adiabatic exponent:

$$\frac{\delta T_{\parallel e}}{T_{0e}} = (G-1)\frac{\delta n_e}{n_{0e}}, \tag{B7}$$

$$G = 2\left[\zeta^2 - \frac{1}{Z'(\zeta)}\right]. \tag{B8}$$

This notation will be useful in our treatment of the tearing mode.

b. Semicollisional limit

In the semicollisional limit, we abandon Eq. (B4) and instead use the linearized Eq. (99) to determine the parallel electron temperature perturbation:

$$\left[\omega + i\kappa_{\|e}k_\|^2(x)\right]\frac{\delta T_{\|e}}{T_{0e}} = \frac{2e}{cm_e}d_e^2\left[k_\|(x)\nabla_\perp^2 A_\| - k_y f''(x)A_\|\right]. \tag{B9}$$

Using Eq. (B2), we rewrite this as

$$\frac{1}{e}\delta T_{\|e} = -\frac{2\omega}{\omega + i\kappa_{\|e}k_\|^2(x)}\frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi. \tag{B10}$$

This can be substituted into Eq. (B3), where also the resistive term is now retained, while the electron inertia must be neglected in comparison (because $\nu_{ei} \gg \omega$).

Recasting Eq. (B10) in the form (B7), we find the effective adiabatic exponent in the semicollisional case

$$G = 1 + \frac{2\omega}{\omega + i\kappa_{\|e}k_\|^2(x)}. \tag{B11}$$

2. Kinetic Alfvén waves

a. Collisionless limit

Let us make a digression and ascertain that the equations we have derived contain the wave dynamics that they are expected to contain. Formally, if we assume $k_z \gg k_y f(x)$, $k_z \gg k_y d_e^2 f''(x)$, and $k_z\partial^2/\partial x^2 \gg k_y f''(x)$, we find ourselves in a homogeneous plasma (all terms containing $f(x)$ can be neglected and $k_\| = k_z$). This means that we can now also Fourier transform in x and, after a few lines of standard algebra, we obtain from Eqs. (B2), (B3), and (B6) the following dispersion relation:

$$\left[\zeta^2 - \frac{\tau}{Z}\frac{k_\perp^2 d_e^2/2}{1-\Gamma_0(k_\perp^2\rho_i^2/2)}\right][1+\zeta Z(\zeta)] = \frac{1}{2}k_\perp^2 d_e^2, \tag{B12}$$

where $k_\perp^2 = k_x^2 + k_y^2$. It is not hard to check that this agrees with the gyrokinetic dispersion relation at low beta derived in Ref. 61 [their Eq. (D17)].

Looking for solutions with $\zeta = \omega/|k_\||v_{\mathrm{th}e} \ll 1$ and using $Z(\zeta) \approx i\sqrt{\pi} - 2\zeta$, we find the well known dispersion relation for Alfvén waves (both MHD and kinetic):

$$\omega = \pm k_\| v_A k_\perp \rho_i \sqrt{\frac{1}{2}\left[\frac{Z}{\tau} + \frac{1}{1-\Gamma_0(k_\perp^2\rho_i^2/2)}\right]}. \tag{B13}$$

The MHD Alfvén waves,

$$\omega = \pm k_\| v_A, \tag{B14}$$

are recovered for $k_\perp\rho_i \ll 1$ (in which case $\Gamma_0 \approx 1 - k_\perp^2\rho_i^2/2$), the kinetic Alfvén waves,

$$\omega = \pm\sqrt{\frac{1}{2}\left(1+\frac{Z}{\tau}\right)}k_\| v_A k_\perp \rho_i, \tag{B15}$$

for $k_\perp\rho_i \gg 1$ ($\Gamma_0 \approx 0$). Finally, the damping rate is found from Eq. (B12) as a small perturbation of ω:

$$\gamma = -|k_\||v_A\frac{k_\perp^2\rho_i^2}{4}\sqrt{\pi\frac{m_e}{m_i}\frac{Z^3}{\tau^2\beta_e}} \tag{B16}$$

(note that this is valid for any $k_\perp\rho_i$). This again is a well known formula for the electron Landau damping of Alfvén waves at low beta [cf. Eq. (63) of Ref. 61 noting that $\beta_e = (Z/\tau)\beta_i$].

Another familiar and useful limit is the long-wavelength, cold-ion approximation, $k_\perp\rho_i \sim \sqrt{\tau} \ll 1$ (already discussed in Sec. III C). Using Eq. (57) in Eq. (B12), we get

$$\left(\zeta^2 - \frac{d_e^2}{2\rho_s^2}\right)[1+\zeta Z(\zeta)] = \frac{1}{2}k_\perp^2 d_e^2. \tag{B17}$$

Assuming again $\zeta \ll 1$, we get for the Alfvén waves in this limit:

$$\omega = \pm k_\| v_A\sqrt{1+k_\perp^2\rho_s^2}, \tag{B18}$$

$$\gamma = -|k_\||v_A\frac{k_\perp^2\rho_s^2}{2}\sqrt{\pi\frac{m_e}{m_i}\frac{Z}{\beta_e}}. \tag{B19}$$

The dispersive kinetic Alfvén waves are believed to play a key role in enabling fast kinetic reconnection,[19,104] so it is important that they are fully retained in our equations.

Note that assuming $\zeta \ll 1$ in the above calculation meant that we effectively adopted an isothermal limit [see Eq. (B6)]. Enforcing this requirement on the frequency [Eq. (B13)] also eliminates the electron scales: $k_\perp d_e \ll 1$ (in the cold-ion limit, this is modified to $d_e \ll \rho_s$). The non-isothermal effects will come in at the electron inertial scale together with the breaking of the flux conservation.

b. Semicollisional limit

Finally, in the semicollisional limit, using Eq. (B10) instead of Eq. (B6), we get the following dispersion relation:

$$\omega(\omega + i\eta k_\perp^2) = k_\|^2 v_A^2 k_\perp^2\rho_i^2 \times\frac{1}{2}\left[\frac{Z}{\tau}\frac{3\omega + i\kappa_{\|e}k_\|^2}{\omega + i\kappa_{\|e}k_\|^2} + \frac{1}{1-\Gamma_0(k_\perp^2\rho_i^2/2)}\right]. \tag{B20}$$

The isothermal limit, Eq. (B13), this time with resistive damping, is recovered for $\kappa_{\|e}k_\|^2 \gg \omega$, or, equivalently, $k_\perp d_e \ll k_\|\lambda_{\mathrm{mfp}e}$.

3. Collisionless tearing mode

Let us now consider the limit opposite to that which produced a homogeneous situation, namely, $k_z \ll k_y f(x)$, so the problem becomes effectively 2D and $k_\|(x) \approx k_y f(x)$. This is

the tearing mode problem. Since we are expecting a (purely) growing mode, let $\omega = i\gamma$. In the collisionless limit, Eqs. (B2), (B3), and (B6) can now be rearranged to read

$$\frac{i\gamma}{k_y f(x) v_{\mathrm{the}}} \frac{Z}{\tau}(1-\hat{\Gamma}_0)\varphi = -\frac{1}{2} d_e^2 \left[\partial_x^2 - k_y^2 - \frac{f''(x)}{f(x)}\right] \frac{v_{\mathrm{the}}}{c} A_\parallel, \tag{B21}$$

$$\frac{i\gamma}{k_y f(x) v_{\mathrm{the}}} \left(1 - d_e^2 \partial_x^2\right) \frac{v_{\mathrm{the}}}{c} A_\parallel = \left[1 + G\frac{Z}{\tau}(1-\hat{\Gamma}_0)\right]\varphi, \tag{B22}$$

where $G(\zeta)$ is the effective adiabatic exponent given by Eq. (B8), $\zeta(x) = i\gamma/|k_y f(x)| v_{\mathrm{the}}$, and we have neglected $d_e^2 k_y^2$ and $d_e^2 f''(x)/f(x)$ compared to unity in Eq. (B22).

As in all tearing-mode calculations, these equations will turn out to have a boundary layer around $x = 0$. The reconnection of the magnetic flux will happen inside this layer, while outside the dynamics will be essentially MHD. The difference between the standard MHD situation[62] and what we have here is that there are two microscales in the problem: ρ_s (or ρ_i) and d_e, so the boundary layer will in fact be made of two nested boundary layers. We will first deal with the outer solution, then give a set of *a priori* estimates of the relevant scales and of the growth rate of the mode that follow from the structure of the problem in the inner region.

a. Outer (MHD) region

Here we seek the solution at the equilibrium scales l, assuming $l \sim [f'(x)/f(x)]^{-1} \sim k_y^{-1} \gg \rho_s \gg d_e$. Accordingly, we can neglect $d_e^2 \partial_x^2 \ll 1$ and $(Z/\tau)(1-\hat{\Gamma}_0) \approx -\rho_s^2(\partial_x^2 - k_y^2) \ll 1$. We are not assuming that ζ is either small or large. Under these approximations, Eq. (B22) becomes simply

$$\frac{i\gamma}{k_y f(x) v_{\mathrm{the}}} \frac{v_{\mathrm{the}}}{c} A_\parallel = \varphi. \tag{B23}$$

Substituting this into Eq. (B21), we get

$$\left[\partial_x^2 - k_y^2 - \frac{f''(x)}{f(x)}\right] A_\parallel = \frac{2\rho_s^2}{d_e^2}\zeta(x)(\partial_x^2 - k_y^2)\zeta(x) A_\parallel. \tag{B24}$$

Now note that, since $\sqrt{2}\rho_s/d_e = v_{\mathrm{the}}/v_A$, we have

$$\frac{\sqrt{2}\rho_s}{d_e}\zeta(x) = \frac{i\gamma}{|k_y f(x)| v_A} = \frac{i\gamma}{\omega_{Ay}}, \tag{B25}$$

where $\omega_{Ay} = |k_y f(x) v_A| = |k_y v_{Ay}|$ is the in-plane Alfvén frequency. Assuming $\gamma \ll \omega_{Ay}$ (tearing mode is sub-Alfvénically slow), we may then neglect the right-hand side of Eq. (B24) and so end up with

$$\partial_x^2 A_\parallel = \left[k_y^2 + \frac{f''(x)}{f(x)}\right] A_\parallel. \tag{B26}$$

This is the standard MHD tearing mode outer-region equation. Its solution depends on the particular choice of $f(x)$, but we will not need its detailed form. Since $\delta B_y = -\partial_x A_\parallel$ must reverse direction at $x = 0$, the sign of $\partial_x A_\parallel$ will be different on either side of $x = 0$, so the derivative will have a discontinuity. This is quantified by

$$\Delta' = \frac{1}{A_\parallel(0)}\left[\partial_x A_\parallel\right]_{-0}^{+0}, \tag{B27}$$

the tearing-mode instability parameter. It is a measure of the singularity developed in the current by the ideal-MHD outer solution as it approaches the reconnection layer. The singularity is resolved by corrections to the ideal Ohm's law (electron inertia in the case we are considering). The solution in the boundary layer around $x = 0$ will have to be matched to the value of Δ'.

b. Inner region: equations

At scales of order ρ_s or smaller, we may expand the in-plane equilibrium profile: $f(x) \approx x f'(0) \equiv x/L_s$. Then $\zeta(x) \approx i\delta/|x|$, where

$$\delta = \frac{\gamma}{k_y v_{\mathrm{the}}} L_s = \frac{\gamma}{k_y v_A} \frac{d_e}{\sqrt{2}\rho_s} L_s. \tag{B28}$$

Since $\partial_x^2 \gg k_y^2, f''(x)/f(x)$ at these scales, Eqs. (B21) and (B22) become

$$\frac{\delta}{x}\frac{Z}{\tau}(1-\hat{\Gamma}_0)\tilde{\varphi} = \frac{1}{2} d_e^2 \partial_x^2 A_\parallel, \tag{B29}$$

$$\frac{\delta}{x}\left(1 - d_e^2\partial_x^2\right) A_\parallel = \left[1 + G\frac{Z}{\tau}(1-\hat{\Gamma}_0)\right]\tilde{\varphi}, \tag{B30}$$

where $\hat{\Gamma}_0 = \hat{\Gamma}_0(\rho_i^2\partial_x^2/2)$, $\tilde{\varphi} = -(c/v_{\mathrm{the}})i\varphi$, and the effective adiabatic exponent G is given by Eq. (B8), which we now rewrite as

$$G\left(\frac{x}{\delta}\right) = -2\left[\frac{\delta^2}{x^2} + \frac{1}{Z'(i\delta/|x|)}\right]. \tag{B31}$$

This function has limiting values $G(0) = 3$, $G(\infty) = 1$ (electrons are isothermal for $x \gg \delta$) and a smooth monotonic transition in between.

In order to benchmark our theory against existing treatments, let us remark that Eqs. (B29) and (B30) can easily be manipulated into a form that coincides with the collisionless tearing mode equations derived by Cowley *et al.*[67] directly from the Vlasov-Maxwell kinetics:

$$\frac{x}{\delta}\left(A_\parallel - \frac{x}{\delta}\tilde{\varphi}\right)\sigma\left(\frac{x}{\delta}\right) = \frac{Z}{\tau}(1-\hat{\Gamma}_0)\tilde{\varphi}, \tag{B32}$$

$$\frac{1}{2}\frac{x}{\delta} d_e^2 \partial_x^2 A_\parallel = \frac{Z}{\tau}(1-\hat{\Gamma}_0)\tilde{\varphi}, \tag{B33}$$

where the "scaled effective conductivity" is

$$\sigma\left(\frac{x}{\delta}\right) = \left[2 + \frac{x^2}{\delta^2} G\left(\frac{x}{\delta}\right)\right]^{-1} = -\frac{1}{2}\frac{\delta^2}{x^2} Z'\left(\frac{i\delta}{|x|}\right). \tag{B34}$$

Equations (B32) and (B33) are Eqs. (26) and (25), respectively, of Ref. 67, specialized to the case of no equilibrium density gradient[112] [note that in Ref. 67, δ is defined in terms of ω, not γ, so to get exactly their equations, one must

replace $\delta \to -i\delta$, $\sigma \to -\sigma$ and restore $\tilde{\varphi} = -(c/v_{the})i\varphi$]. These equations determine the parallel electric field [Eq. (B32)] and the parallel current [Eq. (B33)] in terms of the ion response. This response is, in general, nonlocal, which leads to considerable technical difficulties.[66,67,73,105]

c. Mathematical structure of the problem

Let us introduce a rescaling of the spatial variable $\xi = x/\delta_{in}$, where δ_{in} reflects the width of the inner solution and will be determined later. Equations (B30) and (B29) can be rewritten as follows:

$$\frac{A_\parallel}{\xi} - \frac{\delta_{in}}{\delta}\tilde{\varphi} = \frac{d_e^2}{\delta_{in}^2}\left(1 + \frac{\delta_{in}^2}{2\delta^2}G\xi^2\right)\frac{A_\parallel''}{\xi}, \tag{B35}$$

$$\frac{2\delta\delta_{in}}{d_e^2}\frac{Z}{\tau}(1-\hat{\Gamma}_0)\tilde{\varphi} = \xi A_\parallel'', \tag{B36}$$

where primes denote derivatives with respect to ξ and we have used Eq. (B36) to express the $(Z/\tau)(1-\hat{\Gamma}_0)\tilde{\varphi}$ term in Eq. (B30) in terms of $A_\parallel''$.

Let $\chi(\xi) = \xi A_\parallel' - A_\parallel = \xi^2(A_\parallel/\xi)'$. Then $A_\parallel'' = \chi'/\xi$. Differentiating once Eq. (B35) and expressing $A_\parallel$ in terms of χ, we get

$$\frac{d}{d\xi}\left(\frac{1}{\xi^2} + \frac{\delta_{in}^2}{2\delta^2}G\right)\chi' = \frac{\delta_{in}^2}{d_e^2}\frac{\chi}{\xi^2} - \frac{\delta_{in}^3}{d_e^2\delta}\tilde{\varphi}'. \tag{B37}$$

We now need a closed expression for $\tilde{\varphi}$ in terms of $A_\parallel$, which is where the nonlocality of the ion response—the operator on the left-hand side of Eq. (B36)—presents a technical challenge. The simplest choice is to avoid it by considering the limit of cold ions. It turns out, however, that with no additional trouble, it is possible to accommodate (non-rigorously) the hot-ion limit by using the Padé approximant for the ion response[68]

$$\frac{Z}{\tau}(1-\hat{\Gamma}_0) \approx -\frac{\rho_s^2\partial_x^2}{1-(1/2)\rho_i^2\partial_x^2}. \tag{B38}$$

Then Eq. (B36) becomes

$$-\frac{2\delta\rho_s^2}{d_e^2\delta_{in}}\tilde{\varphi}'' = \chi' - \frac{\rho_i^2}{2\delta_{in}^2}\chi'''. \tag{B39}$$

By integrating this equation once, we work out the $\tilde{\varphi}'$ term in Eq. (B37):

$$-\frac{\delta_{in}^3}{d_e^2\delta}\tilde{\varphi}' = \chi - \chi_0 - \frac{\tau}{Z}\frac{\rho_s^2}{\delta_{in}^2}\chi'', \tag{B40}$$

where χ_0 is a constant of integration, we have used $\rho_i^2/2\rho_s^2 = \tau/Z$, and arranged for the overall coefficient in the left-hand side to be unity by choosing the inner rescaling width

$$\delta_{in} = \left(\sqrt{2}\rho_s\delta\right)^{1/2}. \tag{B41}$$

It remains to substitute Eq. (B40) into Eq. (B37) and arrange terms neatly. The result is

$$\xi^2\frac{d}{d\xi}\left[\frac{1}{\xi^2} + \alpha^2\left(G + \frac{\tau}{Z}\right)\right]\tilde{\chi}' - \left(\xi^2 + \lambda^2\right)\tilde{\chi} = \lambda^2, \tag{B42}$$

where $\tilde{\chi} = -1 + \chi/\chi_0$,

$$\lambda^2 = \frac{\delta_{in}^2}{d_e^2} = \frac{\sqrt{2}\rho_s\delta}{d_e^2} = \frac{\gamma}{k_y v_A}\frac{L_s}{d_e}, \tag{B43}$$

$$\alpha = \frac{\delta_{in}}{\sqrt{2}\delta} = \left(\frac{\rho_s}{\sqrt{2}\delta}\right)^{1/2} = \frac{\rho_s}{d_e\lambda}, \tag{B44}$$

and $G(x/\delta)$ is given by Eq. (B31), with $x/\delta = \sqrt{2}\alpha\xi$.

The above equation is subject to matching with the outer solution. First, using the definition of χ and the property $A_\parallel'' = \chi'/\xi$, we can write

$$A_\parallel = -\chi_0\left[1 + \tilde{\chi} - \xi\int_0^\xi dz\frac{\tilde{\chi}'(z)}{z}\right]. \tag{B45}$$

This is matched to the outer (MHD) solution (see Appendix B 3 a) by identifying the limit $\xi \to \infty$ for the inner solution with $x \to 0$ for the outer solution: this gives $\tilde{\chi}(\xi \to \infty) = 0$, $\chi_0 = -A_\parallel(\xi \to \infty) = -A_\parallel(x \to 0)$ and, from the condition (B27),

$$\int_0^\infty d\xi\frac{\tilde{\chi}'(\xi)}{\xi} = -\frac{1}{2}\Delta'\delta_{in}. \tag{B46}$$

Equations (B42) and (B46) constitute an eigenvalue problem, where λ^2 is the rescaled eigenvalue and α is the parameter that measures the width of the ion layer. Due to the double-layered structure of the inner region, the exact solution requires some work[63,69,73,106] (see Appendix B 4), but all the relevant scalings can be derived in a simple heuristic way. While these scalings are known, it is useful to summarize them in a unified exposition.

d. Ultralow-beta (one-fluid) limit

If $\alpha \ll 1 (\rho_s \ll \delta)$, the ion effects are negligible and our equations turn into the standard tearing mode equations[62] extended to arbitrary Δ' (Refs. 63 and 64) and with resistivity replaced by electron inertia.[71,72]

It is easy to derive the scaling of the growth rate without an exact solution of the equations. From Eq. (B42), $\lambda^2 \sim \tilde{\chi}'/\xi$. The width of the $\tilde{\chi}(\xi)$ function is order unity (i.e., $x \sim \delta_{in}$ in dimensional terms), so the integral in Eq. (B46) is over an interval of order unity. Therefore,

$$\lambda^2 \sim \Delta'\delta_{in}. \tag{B47}$$

Using the definitions (B41) and (B43), one gets the scalings[66,71]

$$\delta \sim \frac{d_e^4\Delta'^2}{\rho_s}, \quad \delta_{in} \sim d_e^2\Delta', \quad \gamma \sim k_y v_A\frac{d_e^3\Delta'^2}{L_s}. \tag{B48}$$

This result breaks down when Δ' is so large that $\Delta'\delta_{in} \sim 1$ or larger, in which case the magnitude of the current is limited by the width of the reconnecting region, δ_{in}. The correct scalings are obtained by replacing Δ' by $1/\delta_{in}$ in Eq. (B47):

$$\lambda^2 \sim 1 \tag{B49}$$

(in other words, instead of estimating the current as $\partial_x^2 A_\parallel \sim (\Delta'/\delta_{\rm in})A_\parallel$, one takes $\partial_x^2 A_\parallel \sim A_\parallel/\delta_{\rm in}^2$). Hence one gets the scalings[65,72]

$$\delta \sim \frac{d_e^2}{\rho_s}, \quad \delta_{\rm in} \sim d_e, \quad \gamma \sim k_y v_A \frac{d_e}{L_s}. \tag{B50}$$

As we explained above, all of this is valid provided $\rho_s \ll \delta$, which amounts to

$$\frac{\rho_s^2}{d_e^2} \sim \beta_e \frac{m_i}{m_e} \ll (d_e\Delta')^2, \tag{B51}$$

for the finite-Δ' scalings (B48) and

$$\frac{\rho_s^2}{d_e^2} \sim \beta_e \frac{m_i}{m_e} \ll 1, \tag{B52}$$

for the infinite-Δ' limit (B50). Either condition can only be satisfied if β_e is very low indeed—not just of order, but much smaller than m_e/m_i [obviously, this is treated as a subsidiary limit within our formal low-beta ordering (12)].

e. Two-fluid case

If $\alpha \gg 1$ or $\rho_s \gg \delta$, the problem contains both ion and electron scales and is no longer mathematically equivalent to the resistive tearing mode [the terms involving α in Eq. (B42) cannot be dropped]. The width of the integration region in Eq. (B46) is now $\xi \sim 1/\alpha$ (or, dimensionally, $x \sim \delta$). Therefore,

$$\lambda^2 \sim \Delta'\delta_{\rm in}\alpha. \tag{B53}$$

This gives immediately the following scalings:[73–76]

$$\delta \sim d_e^2\Delta', \quad \delta_{\rm in} \sim d_e\rho_s^{1/2}\Delta'^{1/2}, \quad \gamma \sim k_y v_A \frac{d_e\rho_s\Delta'}{L_s}. \tag{B54}$$

These scalings are correct provided $\Delta'\delta_{\rm in} \ll 1$, or $\Delta'\rho_s^{1/3}d_e^{2/3} \ll 1$. At larger Δ', again, like in Appendix B 3 d, $\delta_{\rm in}$ limits the magnitude of the current and we replace Δ' by $1/\delta_{\rm in}$ in Eq. (B53):

$$\lambda^2 \sim \alpha, \tag{B55}$$

whence follow the scalings[73,74,76]

$$\delta \sim \frac{d_e^{4/3}}{\rho_s^{1/3}}, \quad \delta_{\rm in} \sim d_e^{2/3}\rho_s^{1/3}, \quad \gamma \sim k_y v_A \frac{d_e^{1/3}\rho_s^{2/3}}{L_s}. \tag{B56}$$

Checking now the condition $\alpha \gg 1$, we find that the finite-Δ' scalings (B54) are valid provided

$$\frac{\rho_s^2}{d_e^2} \sim \beta_e \frac{m_i}{m_e} \gg (d_e\Delta')^2, \tag{B57}$$

and the "infinite-Δ'" ones (B56) hold for

$$\frac{\rho_s^2}{d_e^2} \sim \beta_e \frac{m_i}{m_e} \gg 1. \tag{B58}$$

4. Asymptotic solution for the two-fluid tearing mode

In Refs. 69 and 73, the scalings (B56) were derived in a more mathematical way via a rather involved double-layer matching procedure in wavenumber space under the assumption of isothermal electrons ($G = 1$). There are several other derivations (notably Refs. 74 and 106), also analytically quite cumbersome. Here we give a calculation that recovers the essential result without assuming isothermal electrons and at the minimal analytical cost.

We start by noting that, using Eqs. (B45) and (B46), we may write $A_\parallel$ in a form that is automatically matched to the outer solution:

$$A_\parallel = -\chi_0\left[1 + \tilde{\chi} + \frac{1}{2}\Delta'\delta_{\rm in}\xi + \xi\int_\xi^\infty dz\,\frac{\tilde{\chi}'(z)}{z}\right]. \tag{B59}$$

Our strategy will be first to solve Eq. (B42) in the ion region and use the solution $\tilde{\chi}_i$ in the above formula for $A_\parallel$; then to solve in the electron region and use the solution $\tilde{\chi}_e$ in Eq. (B45); finally to take the large-argument asymptotic of the electron solution $A_{\parallel e}$ and match it with the small-argument asymptotic of the ion solution $A_{\parallel i}$.

a. Ion region

The ion region is $x \sim \rho_s$, or $\xi \sim \alpha \gg 1$. In this limit, the $1/\xi^2$ and $\lambda^2\tilde{\chi}$ terms in Eq. (B42) are negligible (the latter because we are anticipating the ordering $\lambda^2 \sim \alpha$) and $G \approx 1$. Introducing the new variable $z = \xi/\alpha\sqrt{1+\tau/Z} = x/\rho_\tau$, where $\rho_\tau = \rho_s\sqrt{1+\tau/Z}$, we can now write the rest of Eq. (B42) as follows:

$$\tilde{\chi}_i'' - \tilde{\chi}_i = \frac{\tilde{\lambda}^2}{z^2}, \tag{B60}$$

where we have denoted

$$\tilde{\lambda}^2 = \frac{\lambda^2}{\alpha^2(1+\tau/Z)} \ll 1. \tag{B61}$$

Equation (B60) can be solved exactly, subject to the boundary condition $\tilde{\chi}_i(z \to \infty) \to 0$:

$$\tilde{\chi}_i = e^{-z}\left[C_i - \tilde{\lambda}^2\int_{z_0}^z dr\,e^{2r}\int_r^\infty ds\,\frac{e^{-s}}{s^2}\right], \tag{B62}$$

where C_i is a constant of integration and the parameter z_0 in the particular integral can be chosen at will, with the difference absorbed into the homogeneous part of the solution (the constant C_i).

Substituting the solution (B62) into Eq. (B59), taking the limit $z \to 0$ and keeping only the leading-order contributions (in $1/\alpha$) in all terms of the expansion, we get

$$A_{\|i} \approx -\chi_0\left[1 + C_i + \frac{1}{2}\Delta'\delta_{\rm in}\xi - \tilde{\lambda}^2 \ln\xi + \frac{C_i}{\alpha\sqrt{1+\tau/Z}}\xi\ln\xi\right], \tag{B63}$$

where we formally ordered $\Delta'\delta_{\rm in} \sim 1$ and anticipated $C_i \sim 1$ (all of this will be checked *a posteriori*).

b. Electron region

The electron region is $x \sim \delta$ or $\xi \sim 1/\alpha \ll 1$. In this limit, the $\xi^2\tilde{\chi}$ term in Eq. (B42) is negligible and we are left with a homogeneous equation for $\tilde{\chi} + 1 = \chi/\chi_0$. Introducing the new variable $y = \sqrt{2}\alpha\xi = x/\delta$, we get

$$y^2\frac{d}{dy}\left[\frac{1}{y^2} + \frac{1}{2}\left(G(y) + \frac{\tau}{Z}\right)\right]\chi_e' = \frac{\lambda^2}{2\alpha^2}\chi_e. \tag{B64}$$

The right-hand side is, in fact, small, and so we can solve this equation perturbatively, in powers of $\lambda^2/\alpha^2 \sim 1/\alpha$ (cf. Ref. 74). We will see that we need to do this to second order: $\chi_e = \chi_e^{(0)} + \chi_e^{(1)} + \chi_e^{(2)}$. Our boundary condition will be that the current, proportional to χ'/y, must be even as $y \to 0$. We can also let $\chi_e^{(1)}(0) = \chi_e^{(2)}(0) = 0$ without loss of generality, so $\chi_e(0) = \chi_e^{(0)}(0)$.

Integrating Eq. (B64) to zeroth order (right-hand side = 0), we get

$$\frac{\chi_e^{(0)}}{\chi_0} = C_e, \tag{B65}$$

where C_e is a constant of integration. To first order, we obtain, again by direct integration,

$$\frac{\chi_e^{(1)}}{\chi_0} = -\frac{\lambda^2}{2\alpha^2}C_e\int_0^y \frac{du\,u}{1 + (u^2/2)[G(u) + \tau/Z]}. \tag{B66}$$

Finally, to second order,

$$\frac{\chi_e^{(2)}}{\chi_0} = -\frac{\lambda^4}{4\alpha^4}C_e\int_0^y \frac{du\,u^2}{1 + (u^2/2)[G(u) + \tau/Z]} \times \int_0^u \frac{dv}{v^2}\int_0^v \frac{dw\,w}{1 + (w^2/2)[G(w) + \tau/Z]}. \tag{B67}$$

Substituting the solution worked out above into Eq. (B45), taking the limit $y \to \infty$ and again throwing out all terms subdominant in $1/\alpha$, we find

$$A_{\|e} \approx -\chi_0 C_e[1 + I_G\Lambda\xi - \tilde{\lambda}^2\ln\xi + I_G\Lambda\tilde{\lambda}^2\xi\ln\xi], \tag{B68}$$

where $\Lambda = \lambda^2/\alpha\sqrt{1+\tau/Z}$ and

$$I_G = \int_0^\infty \frac{dy\sqrt{1+\tau/Z}/\sqrt{2}}{1 + (y^2/2)[G(y) + \tau/Z]} \tag{B69}$$

is a number of order unity. Had electrons been isothermal ($G = 1$), it would have been $I_G = \pi/2$.

c. The dispersion relation

We now match the asymptotics (B68) and (B63) term by term. The $\ln\xi$ term can be ignored as long as we carry out the matching for $\xi \gg 1$,we can do this because the electron solution is in fact valid beyond the electron region ($\xi \sim 1/\alpha$) all the way through the intermediate region $\xi \sim 1$, or $x \sim \delta_{\rm in}$ (because we have solved to second order in $1/\alpha$).

Matching the remaining three terms allows us to determine the two constants of integration and find the dispersion relation:[69,73]

$$\frac{1}{2}\Delta'\delta_{\rm in} = \frac{I_G\Lambda}{1 - I_G\Lambda^2}. \tag{B70}$$

This indeed has the two limits that we intuited in Appendix B 3 e:

$$\Delta'\delta_{\rm in} \ll 1 \Rightarrow \Lambda \equiv \frac{\lambda^2}{\alpha\sqrt{1+\tau/Z}} = \frac{1}{2I_G}\Delta'\delta_{\rm in}, \tag{B71}$$

$$\Delta'\delta_{\rm in} \gg 1 \Rightarrow \Lambda \equiv \frac{\lambda^2}{\alpha\sqrt{1+\tau/Z}} = \frac{1}{\sqrt{I_G}}, \tag{B72}$$

so we now know not just the scalings but also the numerical prefactors (which depend on the functional form of the effective adiabatic exponent G of the electrons via the constant I_G). Eq. (B70) gives a smooth connection between the two limits.

5. Semicollisional tearing mode

a. Equations

To treat the tearing mode in the semicollisional limit,[66] we must use Eqs. (B2), (B3), and (B10), again taking $k_\|(x) \approx k_y f(x)$. The outer region is still MHD as described in Appendix B 3 a. In the inner region, instead of Eqs. (B29) and (B30), we get (the first equation is unchanged)

$$\frac{\delta}{x}\frac{Z}{\tau}(1 - \hat{\Gamma}_0)\tilde{\varphi} = \frac{1}{2}d_e^2\partial_x^2 A_\|, \tag{B73}$$

$$\frac{\delta}{x}\left(1 - \frac{\eta}{\gamma}\partial_x^2\right)A_\| = \left[1 + G\frac{Z}{\tau}(1 - \hat{\Gamma}_0)\right]\tilde{\varphi}, \tag{B74}$$

$$G\left(\frac{x}{\delta}\right) = \frac{3 + (\kappa_{\|e}\gamma/v_{{\rm the}}^2)(x/\delta)^2}{1 + (\kappa_{\|e}\gamma/v_{{\rm the}}^2)(x/\delta)^2}. \tag{B75}$$

These equations turn out to have a mathematical structure that can be exactly mapped onto the collisionless case.[67] Let us introduce a new scale

$$\delta_\eta = \frac{\delta}{d_e}\left(\frac{\eta}{\gamma}\right)^{1/2} = \frac{L_s^2}{\sqrt{2}\rho_s}\left(\frac{\gamma}{k_y v_A}\right)^{1/2}(k_y L_s S)^{-1/2}, \tag{B76}$$

where $S = v_A L_s/\eta$ is the Lundquist number. Then, denoting $\Phi = (\delta_\eta/\delta)\tilde{\varphi}$, we can recast Eqs. (B73)–(B75) in the following form:

$$\frac{\delta_\eta}{x}\frac{Z}{\tau}(1-\hat{\Gamma}_0)\Phi = \frac{1}{2}\frac{\eta}{\gamma}\partial_x^2 A_\parallel, \tag{B77}$$

$$\frac{\delta_\eta}{x}\left(1-\frac{\eta}{\gamma}\partial_x^2\right)A_\parallel = \left[1+G\frac{Z}{\tau}(1-\hat{\Gamma}_0)\right]\Phi, \tag{B78}$$

$$G\left(\frac{x}{\delta_\eta}\right) = \frac{3+(a/2)(x/\delta_\eta)^2}{1+(a/2)(x/\delta_\eta)^2}, \tag{B79}$$

where $a = 2\eta\kappa_{\|e}/d_e^2 v_{\mathrm{the}}^2$. For the collision operator we chose in Sec. III D, $\eta = \nu_{ei}d_e^2$, $\kappa_{\|e} = v_{\mathrm{the}}^2/2\nu_{ei}$, and so $a=1$; the isothermal closure would formally correspond to $a=\infty$. Comparing Eqs. (B77)–(B79) with Eqs. (B29)–(B31), we see that all results obtained for the collisionless case can be converted into analogous results for the semicollisional case by mapping

$$\delta \to \delta_\eta, \quad d_e^2 \to \frac{\eta}{\gamma} \tag{B80}$$

and using Eq. (B79) instead of Eq. (B31) for the effective adiabatic exponent of the electrons.

Again, these equations can be manipulated into a form derived in Ref. 67. Proceeding analogously to the way we did in Appendix B 3 b, we get

$$\frac{x}{\delta_\eta}\left(A_\parallel - \frac{x}{\delta_\eta}\Phi\right)\sigma\left(\frac{x}{\delta_\eta}\right) = \frac{Z}{\tau}(1-\hat{\Gamma}_0)\Phi, \tag{B81}$$

$$\frac{1}{2}\frac{x}{\delta_\eta}\left(\frac{\delta_\eta^2}{\delta^2}d_e^2\right)\partial_x^2 A_\parallel = \frac{Z}{\tau}(1-\hat{\Gamma}_0)\Phi, \tag{B82}$$

where the "scaled effective conductivity" is

$$\sigma\left(\frac{x}{\delta_\eta}\right) = \frac{1+(a/2)(x/\delta_\eta)^2}{2+(3+a)(x/\delta_\eta)^2+(a/2)(x/\delta_\eta)^4}. \tag{B83}$$

Equations (B81) and (B82) are Eqs. (76) and (78), respectively, of Ref. 67, again in the special case of no equilibrium density gradient (to recover their equations exactly, replace $\delta \to -i\delta$, $\delta_\eta \to \delta_\eta/\sqrt{2}$, and $\sigma \to \sigma/2$).

Using the mapping (B80), we conclude that the general tearing mode equations (B42) and (B46), derived in Appendix B 3 c, now hold with

$$\delta_{\mathrm{in}} = \left(\sqrt{2}\rho_s\delta_\eta\right)^{1/2} = L_s\left(\frac{\gamma}{k_y v_A}\right)^{1/4}(k_y L_s S)^{-1/4}, \tag{B84}$$

$$\lambda^2 = \frac{\sqrt{2}\rho_s\delta_\eta}{\eta/\gamma} = \left(\frac{\gamma}{k_y v_A}\right)^{3/2}(k_y L_s S)^{1/2}, \tag{B85}$$

$$\alpha = \frac{\delta_{\mathrm{in}}}{\sqrt{2}\delta_\eta} = \frac{\rho_s}{\sqrt{2}\delta_\eta} = \frac{\rho_s}{L_s}\left(\frac{\gamma}{k_y v_A}\right)^{-1/4}(k_y L_s S)^{1/4}, \tag{B86}$$

and $G(\sqrt{2}\alpha\xi)$ given by Eq. (B79). All the same mathematical considerations apply, with the (qualitatively inconsequential) exception that the functional form of the effective adiabatic exponent is different.

We conclude by summarizing the heuristically obtainable scalings for the semicollisional case—these can be read off from the results of Appendices (B3) and (B4) with the aid of the definitions (B84)–(B86). From this point on, all lengths are normalized by L_s.

b. Resistive MHD (one-fluid) limit

Proceeding analogously to the case considered in Appendix B 3 d, we assume $\alpha \ll 1$, or $\rho_s \ll \delta_\eta$, and recover the classic resistive-MHD tearing mode.[62–64] First, at finite Δ', Eq. (B47) with the definitions (B84) and (B85) gives the scalings[62]

$$\delta_\eta \sim \Delta'^{2/5}\rho_s^{-1}(k_y S)^{-4/5}, \tag{B87}$$

$$\delta_{\mathrm{in}} \sim \Delta'^{1/5}(k_y S)^{-2/5}, \tag{B88}$$

$$\frac{\gamma}{k_y v_A} \sim \Delta'^{4/5}(k_y S)^{-3/5}. \tag{B89}$$

The reconnection layer's width is $\delta_{\mathrm{in}} \ll \delta_\eta$ (because $\alpha \ll 1$) and the condition for the above scalings to apply is $\Delta'\delta_{\mathrm{in}} \ll 1$, which translates into

$$\Delta' \ll (k_y S)^{1/3}. \tag{B90}$$

When this is broken, one gets the "infinite-Δ'" scaling (B49), whence follow the scalings[63,65]

$$\delta_\eta \sim \rho_s^{-1}(k_y S)^{-2/3}, \tag{B91}$$

$$\delta_{\mathrm{in}} \sim (k_y S)^{-1/3}, \tag{B92}$$

$$\frac{\gamma}{k_y v_A} \sim (k_y S)^{-1/3}. \tag{B93}$$

The resistive MHD results are valid provided $\alpha \ll 1$, which imposes an upper bound on the ion scale

$$\rho_s \ll \Delta'^{1/5}(k_y S)^{-2/5} \tag{B94}$$

for the finite-Δ' scalings and

$$\rho_s \ll (k_y S)^{-1/3} \tag{B95}$$

in the "infinite-Δ'" limit.

c. Two-fluid case

When the ion scale is sufficiently large ($\alpha \gg 1$), two-fluid effects become important, similarly to the case considered in Appendix B 3 e. Using Eq. (B53) and the definitions (B84)–(B86), one gets the scalings[66,68]

$$\delta_\eta \sim \Delta'^{1/3}\rho_s^{-2/3}(k_y S)^{-2/3}, \tag{B96}$$

$$\delta_{\mathrm{in}} \sim (\Delta'\rho_s)^{1/6}(k_y S)^{-1/3}, \tag{B97}$$

$$\frac{\gamma}{k_y v_A} \sim (\Delta'\rho_s)^{2/3}(k_y S)^{-1/3}. \tag{B98}$$

These scalings hold provided $\Delta'\delta_{\mathrm{in}} \ll 1$, or

$$\Delta' \ll \rho_s^{-1/7}(k_y S)^{2/7}. \tag{B99}$$

At larger Δ', Eq. (B55) must be used, whence follow the scalings[65,69,70]

$$\delta_\eta \sim \rho_s^{-5/7}(k_y S)^{-4/7}, \tag{B100}$$

$$\delta_{\rm in} \sim \rho_s^{1/7}(k_y S)^{-2/7}, \tag{B101}$$

$$\frac{\gamma}{k_y v_A} \sim \rho_s^{4/7}(k_y S)^{-1/7}. \tag{B102}$$

In both cases, the width of the reconnection layer is $\delta_\eta \ll \delta_{\rm in}$ (because $\alpha \gg 1$), which holds if

$$\rho_s \gg \Delta'^{1/5}(k_y S)^{-2/5} \tag{B103}$$

for the finite-Δ' scalings and

$$\rho_s \gg (k_y S)^{-1/3}, \tag{B104}$$

in the "infinite-Δ'" limit.

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[108]It is probably also essential, for a similar reason, that the model be three-dimensional; restriction to exact two-dimensionality imposes many additional conservation laws (see Appendix A).
[109]It can also be thought to describe the nonlinear situation, rather than the linearized one, if Eq. (98) is first transformed into the Lagragian frame moving with the plasma and the parallel coordinate measured along the exact field line. This, however, is not an essential point as the phase mixing mechanism we are considering is fundamentally linear.
[110]The spectrum with the same m dependence as Eq. (105) was first derived in Ref. 107 for electrostatic ITG turbulence in tokamaks, although, unlike those authors, we have not invoked any nonlinear dynamics in its derivation.
[111]Note, however, the claim in Ref. 54 that the inclusion of electron kinetics alleviates somewhat the problem of formation of singularities in their 2D Hamiltonian fluid-kinetic model.
[112]This reduction is accomplished by setting the electron density gradient $1/a$ and, therefore, $\omega_* = k_y c T_0 e/eB_0 a$ to zero in all of their equations except Eq. (25), where $(\omega_*/\omega)^2/\beta_p = d_e^2/2\delta^2$, because the density gradient in ω_* cancels with the one in $\beta_p = (4\pi n_{0e} T_{0e}/B_0^2) L_s^2/a^2$.