Resistive wall mode stabilization in toroidal geometry

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The possibility of stabilizing ideal magnetohydrodynamical (MHD) instabilities by resistive walls and slow plasma rotation (rotation frequencies comparable to resistive tearing growth rates) was proposed recently by Finn [Phys. Plasmas 2, 3782 (1995)] on the basis of cylindrical theory. In the present paper we analyze toroidal effects (pressure gradients and favorable averaged curvature) [Glasser et al., Phys. Fluids 18, 875 (1975)] on this “resistive window.” It is found that in toroidal geometry the resistive window for the distance of the wall from the plasma scales as $S^{-1/3}$ ($S$ is the ratio of resistive to Alfvénic time scales) and thus becomes very small in large tokamaks. Other differences between toroidal and cylindrical theories of resistive wall mode stability are discussed.

In order to reach significant beta at a high bootstrap fraction, the tokamak is dependent on wall stabilization of ideal MHD (magnetohydrodynamical) modes with low toroidal mode numbers. Theoretical analysis has shown that attractive equilibria with hollow current profiles and excellent transport properties can be found if a perfectly conducting wall is placed within about 30% of the minor radius from the plasma boundary. In the absence of conducting walls, tokamaks at conventional aspect ratios and shaping are limited to unattractively low beta if a high fraction of bootstrap current is required.

Experimental work has shown that plasma pressures, well above the ideal-MHD limit in the absence of a wall, can be stably confined, and this has been attributed to stabilization by resistive walls. Theories of resistive wall stabilization rely on plasma rotation. An important question, that has not yet been settled, is how rapid the rotation needs to be in order to give a substantially increased beta limit, so that an economically attractive “advanced” tokamak reactor could be conceived. Numerical analyses with toroidal MHD codes predict that the required rotation speeds are at a few percent of the Alfvén speed; however, experiments suggested that significantly slower rotation may be sufficient. This led to several attempts to explain stabilization at slower rotation. Most of the attempted explanations rely on some dissipation mechanism, such as resistivity and viscosity at the plasma edge or wave–particle resonance, but also the effects of partial walls and non-linear island formation have been discussed.

An interesting explanation was proposed by Finn, who found a stabilization mechanism with “resistive” flow speeds $O(v_A S^{-1/3})$. Unlike the ideal-MHD stabilization, where the optimum wall position is just inside the marginal position of a perfect wall against ideal MHD modes, Finn’s mechanism requires a closer-fitting wall, located just inside the marginal position for resistive tearing modes. It was later shown that Finn’s resistive window is quite small in terms of wall location and rotation velocity.

In the present paper we draw attention to toroidal effects on the resistive stabilization. We account for favorable averaged curvature, represented by a negative resistive interchange parameter $D_R < 0$, and model the layer physics by the dispersion relation of Glasser, Greene, and Johnson. An important consequence is that the width of the resistive window for the wall position scales as $(D_R)^{-5/6} S^{-1/3}$, and becomes vanishingly small for large tokamaks. Other significant differences between toroidal and cylindrical theories of resistive wall modes are discussed at the end of the paper.

To derive an eigenvalue problem including favorable curvature, we consider a case with a single resonant surface. Toroidal effects will be included only in the layer and the coupling between different poloidal Fourier components will not be analyzed (although this is the main source of free energy in the external region). The global stability will be
described similarly as in cylindrical theory\textsuperscript{10}: \[ \Delta_{\text{ext}} = a \Delta_{\text{int}}(\gamma) = (\overline{\sigma}_0 + \gamma \tau_r \overline{d}_1) (\psi_0 + \gamma \tau_r \psi_1), \] where \( \delta_0 \) and \( \delta_1 \) denote jumps in \( \psi' \) while \( \psi_0 \) and \( \psi_1 \) are the values (all evaluated at the resonant surface) of certain solutions\textsuperscript{22} to the ideal-MHD force balance equation without a wall (subscript 0) and with a perfect wall (subscript 1). Here \( \tau_w \) is the resistive wall time, and \( \gamma \) is the growth rate in the laboratory system. At the resonant surface, the growth rate acquires an imaginary part, given by the local plasma rotation frequency \( \Omega \), and the eigenvalue problem reads as

\[ \frac{\delta_0 + \gamma \tau_w \delta_1}{\psi_0 + \gamma \tau_w \psi_1} = f(\gamma - i \Omega), \] (1)

where \( f \) denotes \( \Delta_{\text{int}} = a \Delta' \) from the resistive layer. The dispersion relation of Glasser, Greene, and Johnson\textsuperscript{23} gives

\[ f(\gamma) = (\tau_L)^{3/4}(1 + \frac{D}{(\gamma \tau_L)^{3/2}}). \] (2)

Here \( \tau_L = S^{5/2} \tau_A \) is the resistive layer time (for negligible plasma pressure). Furthermore \( D = -D_R S^{5/2} \) is a normalized resistive interchange parameter, which is generally larger in large tokamaks. (Numerical constants of order unity in Ref. 23 have been left out, but could be incorporated in the definitions of \( S \) and \( D_R \).)

We concentrate on tokamaks with favorable curvature \((q > 1 + \text{corrections for shear and shaping})\)\textsuperscript{21,24} so that \( D_R \) is negative and \( D \) positive, and assume ideal instability in the absence of a wall \([\psi_0 < 0, \sigma_0 = 0(1) > 0]\) and ideal stability with a perfectly conducting wall \([\psi_1 > 0]\).

It is convenient to scale the variables by introducing the resistive interchange scaling,

\[ \gamma = \tau_L D^{2/5}, \quad f = D^{6/5}(\gamma^{3/4} + \gamma^{1/4}) = D^{6/5} \hat{f}. \] (3)

Other variables with overcarets are defined similarly, so that delta primes are scaled as \( \hat{\Delta} = \Delta / D^{5/6} \) and the resistive wall time as \( \tau_w = \tau_L D^{2/5} \tau_r = (-D_R)^{2/5} S^{-1/5} \tau_r / \tau_A \). Equation (3) indicates that the thresholds for \( \Delta_{\text{int}} = \delta_0 / \psi_0 \) and \( \Delta_1 = \delta_1 / \psi_1 \) must be of the order \( D^{6/6} = (-D_R)^{5/6} S^{1/2} \) (as in the Glasser et al. threshold\textsuperscript{23}). Our main point is that since the thresholds for the delta primes are typically very large, the stabilization by resistive flows can be effective only when the system is very near ideal marginality. This is similar to what happens to resistive tearing modes in toroidal geometry.

The stability properties of the eigenvalue problem (1)–(2) can be analyzed straightforwardly in the limit of long wall time, \( \tau_w \gg 1 \). Then, the resistive wall mode will have \(|\gamma| \gg |\Omega|\) so that one can approximate \( \gamma - i \Omega \approx -i \Omega \) in the internal \( \Delta' \). Equation (1) then becomes an algebraic equation for \( \gamma \). It is instructive to consider Nyquist plots, i.e., curves of \( \Delta_{\text{ext}} - \Delta_{\text{int}} \) as \( \gamma \) encircles the unstable half-plane in the positive sense, and use the Cauchy theorem of phase variation to count the number of instabilities. Under the assumption of a long wall time, one can reduce the Nyquist diagrams for (1) to the circles produced by \( \Delta_{\text{ext}}(i \omega) \) when \(-\infty < \omega < \infty\) for a fixed rotation frequency \( \Omega \). When \( \phi_0 < 0 \) \( < \psi_1, \Delta_{\text{ext}}(\gamma) \) has one pole in the unstable half-plane, and therefore the \( \Delta'_\text{ext} - \Delta'_\text{int} \) contour must encircle the origin once in the negative sense for the system to be stable. Equivalently, the point \( f(-i \Omega) \) must lie inside the circle \( \Delta_{\text{ext}}(\gamma) \) as \( \gamma = i \omega \) traverses the imaginary axis. Using the qualitative nature of \( f(-i \Omega) \) we see that marginal stability occurs when the two curves \( \Delta_{\text{ext}}(i \omega) \) and \( \Delta_{\text{int}}(-i \Omega) \) touch the manner shown in Fig. 1. The point where the \( f(\pm i \Omega) \) curves cross the real axis gives the critical \( \Delta' \) of Glasser, Greene, and Johnson \((\hat{\Delta}_G = \Delta(\pm i \omega) = [\tan(\pi/8)]^{5/6} [\sqrt{2} \sin(\pi/8)] \approx 0.886 \) with \( \hat{\omega}_G = [\tan(\pi/8)]^{2/3} \).

Now, assume that Eq. (1) is satisfied for some \( \gamma = i \omega \) and \( \Omega \). In a marginal case such as in Fig. 1, where the curves \( \Delta_{\text{int}}(i \omega) \) and \( f(-i \Omega) \) touch, the derivatives of the two sides of (1) with respect to the (two different) frequencies must have the same phase, so that

\[ \frac{\delta_1 \psi_0 - \delta_0 \psi_1}{(\psi_0 + i \omega \tau_w \psi_1)} = R f', \]

with \( R \) positive and real. This implies that

\[ \omega \tau_w \psi_1 / \psi_0 = \text{Im}(z) / \text{Re}(z), \quad z = (-1f')^{1/2}. \] (4)

With this solution for \( \gamma \), the real and imaginary parts of (1) give the critical delta primes,

\[ \Delta_{\text{oc}} = \frac{\delta_1}{\psi_0} = \frac{\text{Re}(fz)}{\text{Re}(z)}, \quad \Delta_{1c} = \frac{\delta_1}{\psi_1} = \frac{\text{Im}(fz)}{\text{Im}(z)} . \] (5)

The resistive layer expression for \( f \) and \( f' \) can be evaluated at any rotation frequency \( \Omega \) to give a pair of marginal \( \hat{\Delta}_{0c}(\hat{\Omega}) \) and \( \hat{\Delta}_{1c}(\hat{\Omega}) \). However, the rotation frequency must exceed \( \hat{\omega}_G \). As \( |\hat{\Omega}| < \hat{\omega}_G, \Delta_{1c} < \Delta_G, \) and, for \( \hat{\Delta}_1 > \hat{\Delta}_G, \) the system...
is unstable, even with an ideal wall.) For the case considered in Fig. 1, \( \Delta_{1c} \) is a lower bound for stability, while \( \Delta_{0e} \) is an upper bound. (It is also possible to invert the roles of \( \Delta_0 \) and \( \Delta_1 \), but since we are considering \( \Delta_0 < 0 \), the stable region is then restricted to large negative \( \Delta_1 \).

The result (5) is displayed in Fig. 2 as a diagram of \( \Delta_1 \) vs \( \Delta_0 \). The region marked “stabilizable” is stable if the rotation frequency is in the appropriate range, typically connected with the resistive interchange ordering: \( \Omega \sim \omega_A S^{-1/3}(-DR)^{2/3} \). Stability also requires a sufficiently long wall time, as will be discussed shortly.

As \( D^{5/6} \) is typically large, the global stability problem will, typically, give numerically small \( \Delta_0 \) and \( \Delta_1 \) unless the system is near marginal in ideal MHD. Figure 2 shows that the region around the origin cannot be stabilized, and, consequently, the resistive window is very narrow. There are two regions of interest. One is \( \Delta_0 = 0 \), which means robust ideal instability in the absence of a wall. In this case, resistive stabilization is possible for \( 0.800 < \Delta_1 < 0.886 \), which means that the wall must be within a distance of order \( a \) \( (-D_R)^{-5/6}S^{1/3} \) from the ideal marginal position. The other case of interest is when the wall is not very close to the ideal-MHD marginal position, so that \( \Delta_1 \approx 0 \). In this case, resistive stabilization requires \( \Delta_0 < -14.51 \), that is, only a very weak ideal instability can be wall stabilized.

Concerning the influence of the wall time, we note that the preceding analytical treatment assumes a long wall time, but the wall time was never explicitly used in the calculation. In the stabilizable region, we can assert that there are two rotation frequencies for which one solution of the dispersion relation is marginally stable, and this fact is independent of the value of \( \tau_m \). For sufficiently long wall times, these marginal solutions are necessarily connected to the resistive wall mode, and the wall mode is then stabilized between the two rotation frequencies. However, for short wall times there are other possibilities.

To see the implications of a finite wall time, we have solved the dispersion relation numerically, following roots as the rotation frequency increases from zero to a large value. This study confirms the analytical results if the wall time is long. Under this condition, the resistive wall root of the dispersion relation preserves its identity as the rotation frequency changes. Such a case is shown in Fig. 3(a), where \( \tau_m = 30 \), \( \delta_0 = \delta_1 \), \( \Delta_0 = -2 \), \( \Delta_1 = 0.7 \), near the stability boundary in Fig. 2. If we decrease the wall time, the wall mode will eventually switch over to a “plasma” mode as the rotation frequency varies. For the case in Fig. 3, the transition occurs at \( \tau_m = 11 \). Figure 3(b) shows the roots in the complex plane for wall times just below the transition, \( \tau_m = 10 \). In this case there is still a stable window in rotation frequency, because root 1 (the wall mode for \( |\Omega| \leq 1.3 \)) leaves the unstable half-plane before root 2 (wall mode for \( |\Omega| \geq 1.3 \)) enters. This holds true for \( 4 \leq \tau_m \leq 11 \), but when the wall time is too short, \( \tau_m \leq 4 \), the stable window disappears completely. The window disappears more easily (i.e., for longer wall times) the larger \( \Delta_1 \) is. As might be expected, stability is lost, even for very long wall times, when \( \Delta_1 \) approaches the Glasser critical \( \Delta' \).

The numerical results of solving the full eigenvalue equation show that a finite wall time erodes the top part of the stabilizable region in Fig. 2. Consequently, the stability diagram applies to the most optimistic case of very long wall times.

In summary, we have introduced toroidal effects into the layer physics to study the stability of resistive wall modes. An important result is that favorable averaged curvature and pressure practically eliminate the possibility for stabilization at resistive flow velocities. The window in wall position (i.e., the distance between the wall and the plasma) becomes very small \( \left( O(a (-D_R)^{-5/6}S^{-1/3}) \right) \) at high \( S \), so that the resistive stabilization mechanism, proposed on the basis of cylindrical theory, is irrelevant to large tokamaks.

Although the cylindrical theory also predicts a rather small resistive window, the overall picture is significantly changed by toroidicity. In cylindrical theory, the “resistive stability window” occurs when the wall is just inside the marginal position (of an ideal wall) for tearing modes, far inside the marginal position for ideal MHD modes. When the wall is between the marginal positions for tearing and ideal modes, cylindrical theory predicts an unconditionally unstable tearing mode rotating with the plasma. In toroidal theory, this tearing mode is stabilized by favorable averaged curvature. Furthermore, the “resistive window” occurs when the wall is close to the marginal position for ideal MHD modes. At this position, a low rotation frequency, of order \( \omega_A (-D_R)^{-5/6}S^{1/3} \), is sufficient also for ideal
stabilization.\textsuperscript{12,13,22} Therefore, the resistive window not only shrinks but also turns into a slight modification of the ideal stability window.] For walls closer than this distance, the tearing mode is eliminated by toroidality and the ideal-MHD resistive mode can be stabilized by a rotation frequency in the ideal range. The necessary rotation frequency increases if the wall is moved inward.

As a consequence, the cylindrical and toroidal theories give entirely different predictions for the necessary wall position. The cylindrical theory predicts stability of both ideal and resistive modes, only when the wall is in the small “resistive window” just inside the marginal position for tearing. In toroidal MHD, there is a wide range of wall positions where both ideal and resistive modes can be stable for an ideal-MHD rotation velocity. The wall position that requires the lowest rotation speed for stabilization is just inside the ideal marginal position (of an ideal wall).\textsuperscript{12,13,22} Our toroidal results are consistent with those from the resistive-MHD code MARS.\textsuperscript{13,17} They can be seen as examples of the near-elimination of resistive tearing modes by favorable curvature in tokamaks.\textsuperscript{23}

Toroidal layer physics thus removes one of the prime candidates for explaining resistive wall stabilization with very small rotation speeds. We would like to mention that recent experimental results\textsuperscript{25} give higher values for required rotation speed than previously reported. It appears possible, although by no means conclusively established, that ideal MHD in toroidal geometry gives a sufficiently good description of wall stabilization. It is also possible that other physics effects, such as, e.g., wave–particle resonance, two-fluid,\textsuperscript{26} and neoclassical effects are needed to give reliable predictions.

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