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The influence of fast ions on the magnetohydrodynamic stability of negative shear profiles

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The influence of energetic ions on the stability of ideal double kink modes in a tokamak plasma with negative magnetic shear is investigated. It is found that the fast ions play a similar role as for the ordinary \( m = n = 1 \) internal kink. In particular, phenomena analogous to sawtooth stabilization and fishbone excitation are possible. [S1070-664X(97)00805-7]

I. INTRODUCTION

Several tokamaks have recently observed substantially improved confinement in discharges with hollow current profiles\(^1\)\(^-\)\(^5\) where the safety factor \( q(r) \) is a non-monotonic function of the radius \( r \), usually with a single off-axis minimum \( q_{min} \). If \( q_{min} \) is less than, but close to, a rational number, \( q_{min} = m/n \), a magnetohydrodynamic (MHD) mode with poloidal and toroidal mode numbers \( m \) and \( n \), respectively, may become unstable and limit the achievable \( \beta \) of the plasma. An analytical study of the stability of these “double kink” modes was recently published.\(^9\) The instability is driven by the combined action of the pressure gradient and unfavorable magnetic curvature. Mode-coupling to poloidal side bands plays a crucial role, and the mode structure was found to be concentrated in the region where \( q = m/n \). Experimentally, evidence of similar modes (or rather their resistive, reconnecting counterparts) with \( m/n = 2/1 \) has been observed in the form of off-axis sawteeth localized in the region around the \( q = 2 \) surface.\(^3\)\(^,\)\(^8\)

In the present work, we investigate the influence of fast ions, such as fusion-produced alpha particles and neutral-beam injected ions, on the double kink mode. Fast ions are known to interact with the \( m = n = 1 \) internal kink in plasmas with \( q(0) < 1 \),\(^9\)\(^-\)\(^12\) being able to both stabilize sawteeth and to excite so-called fishbone oscillations. Here, we show that energetic ions play a similar role for the stability of the double kink.

The dispersion relation has the form

\[
\delta I + \delta W_c + \delta W_f + \delta W_b = 0, \tag{1}
\]

where \( \delta I \) is the kinetic energy, shown in Ref. 6 to be equal to [see Eq. (31) in that paper]

\[
\delta I = -\frac{i \omega}{\omega_h} \frac{2 \pi B^2}{\mu_0 m q R} \sqrt{1 + 2 q'(r_1)^2 s_1 + r_2^2 s_2} \xi_0^2.
\]

Here \( B \) is the magnetic field strength, \( R \) the major radius of the torus, \( r_1 \) and \( r_2 \) the minor radii of the rational surfaces where \( q = m/n \), \( \xi_0 \) the jump in the radial component of the plasma displacement across these surfaces, \( s_1 \) and \( s_2 \) the magnetic shear, \( q'(r) / q(r) \), at these locations, \( \omega \) the mode frequency, and \( \omega_h = B/R (\mu_0 \rho)^{1/2} \) with \( \rho \) the plasma density is the Alfven frequency. The next term, \( \delta W_c \), in the dispersion relation is the usual MHD potential energy of the bulk plasma, and the last two terms represent the energy associated with the “fluid” and “kinetic” response of the fast ions.

Of course, if there are no fast ions \( \delta W_f = \delta W_b = 0 \) and the plasma is unstable if \( \delta W_c < 0 \) since the growth rate becomes positive, \( \gamma = \text{Im} \omega > 0 \), from (1).

If isotropic fast ions are present, it is natural to include their fluid response \( \delta W_f \) in the MHD potential energy of the background plasma by writing \( \delta W_{MHD} = \delta W_c + \delta W_f \). Instability then occurs if \( \delta W_{MHD} + \text{Re} \delta W_b < 0 \). Thus, if \( \text{Re} \delta W_b \) is positive, which is commonly the case for the usual \( m = n = 1 \) internal kink, the plasma is stable above the ideal MHD threshold. We show that this is the case also for the double kink mode. A phenomenon analogous to sawtooth stabilization thus appears possible.

Neutral-beam injected ions are known not only to stabilize sawteeth, but also to excite finite-frequency fishbone oscillations.\(^13\)\(^,\)\(^14\) This occurs when the total hot-ion energy \( \delta W_b = \delta W_f + \delta W_b \) has an imaginary part as well as a large negative real part. The real part of the dispersion relation (1) then indicates the existence of a mode with a growth rate,

\[
\gamma \propto -\delta W_c - \text{Re} \delta W_b.
\]

that can be positive in equilibria which are below the ideal MHD stability limit, while the imaginary part of the dispersion relation determines the real part of the mode frequency. The latter is usually found to be of the order of the fast-ion toroidal precession frequency (or the ion diamagnetic frequency if such effects are retained).\(^15\) Again, this is a well known feature of the usual \( m = n = 1 \) kink and we show that it carries over naturally to the double kink.

A technical difference between the internal \( m = n = 1 \) mode and the double kink is that the latter is more likely to have a narrow mode structure \( \Delta_m \), comparable to the banana orbit width \( \Delta_b \), or even the Larmor radius, of the fast ions. Finite orbit-width effects are therefore important. Accordingly, we carry out the analysis in both the limits \( \Delta_m \ll \Delta_b \) and \( \Delta_m \approx \Delta_b \), and allow the fast-ion Larmor radius to be comparable to the mode width. The necessary formalism for solving the drift kinetic equation is developed in Sec. II. In the following two sections the low-frequency and finite-frequency responses of the fast ions are calculated and in the last section our conclusions are summarized.
II. FAST-ION RESPONSE

To determine the response of energetic ions to the magnetic field perturbation associated with the instability, it is necessary to solve the kinetic (Vlasov) equation for the fast-ion distribution function and to calculate the energy contained in the perturbation thereof. This was accomplished in a recent paper by Porcelli et al., whose analysis allows for arbitrary orbit width, but assumes a small Larmor radius. For the sake of completeness, we reproduce a shorter version of their calculation here, generalizing it to allow for an arbitrary Larmor radius.

The unperturbed distribution function \( f \) is a function of constants of motion in the unperturbed magnetic field, \( f = f(\mathbf{I}) \), i.e., \( \mathbf{I} = (L, \mu, p_\varphi) \), where \( L = M v^2 / 2 \) is the kinetic energy, \( \mu = M v^2 / 2B \) the magnetic moment, and \( p_\varphi = M R v_\varphi + Z e \psi \) is the toroidal canonical momentum. Here, \( M \) is the mass, \( Z e \) the charge, \( R \) the major radius, and \( \mathbf{B} = \mathbf{F} \nabla \psi + \nabla \psi \times \nabla \psi \) the magnetic field, with \( \psi \) the toroidal angle and \( \varphi \) the poloidal flux function.

The perturbation of the distribution function is, of course,

\[
\delta f = - \partial \mathbf{I} \cdot \frac{\partial f}{\partial \mathbf{I}},
\]

where \( \partial \mathbf{I} \) denotes the perturbation of the constants of motion,

\[
\delta \mathbf{I} = Z e \int_{-\infty}^{t} \mathbf{v} \cdot \partial \mathbf{E} \, dt' = - \int_{-\infty}^{t} \frac{\partial \mathbf{L}}{\partial t'} \, dt',
\]

\[
\delta p_\varphi = \int_{-\infty}^{t} \frac{\partial \mathbf{L}}{\partial \varphi} \, dt' - Z e R \delta A_\varphi,
\]

\[
\delta \mu = \mu \frac{\partial \mathbf{B}}{B}.
\]

Here, \( L = M v^2 / 2 + Z e \delta \mathbf{v} \cdot \mathbf{B} \) is the Lagrangian, with \( \delta \mathbf{A} \) the magnetic potential, whose perturbed part we take to vary as \( e^{-i (\omega t + \varphi \psi)} \). It is assumed that there is no equilibrium electric field, and that \( \delta \mathbf{A} \) is chosen with such a gauge that the electric field associated with the instability is \( \delta \mathbf{E} = - \partial \delta \mathbf{A} / \partial t \).

The increase in energy, \( \Delta \mathcal{E} \), in one gyroperiod \( 2 \pi / \Omega = 2 \pi M \omega / Z e B \), is evidently given by

\[
\Delta \mathcal{E} = \frac{\partial \mathbf{I}}{Z e} \cdot \delta \mathbf{E} \cdot d \mathbf{I} + \int \delta \mathbf{E} \cdot d \mathbf{I}
\]

\[
= \int \int \frac{\partial \mathbf{B}}{\partial t} d \mathbf{S} + \Delta \mathbf{R} \cdot \delta \mathbf{E},
\]

where \( C \) denotes the particle trajectory, comprising one Larmor gyration superimposed on the small displacement of the guiding center \( \Delta \mathbf{R} = 2 \pi \mathbf{v}_d / \Omega \) caused by the magnetic drift,

\[
\mathbf{v}_d = \frac{1}{Z e B} \mathbf{b} \times (\mathbf{\mu} \nabla B + M v^2 \mathbf{\kappa}).
\]

Here, \( \mathbf{b} = \mathbf{B} / B \) and \( \mathbf{\kappa} = (\mathbf{b} \cdot \nabla) \mathbf{b} \) is the magnetic field curvature. The curve \( C - \Delta \mathbf{R} \) is closed (cf. Fig. 1), and we have applied Stokes’s theorem to the curve integral, converting it to a surface integral measuring the total magnetic flux linked by the curve by noting that \( \nabla \times \delta \mathbf{E} = - \partial \mathbf{B} / \partial t \). It follows that the gyro-averaged rate of change in the energy is equal to

\[
Z e \mathbf{v} \cdot \delta \mathbf{E} = \mu \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}_d \cdot \delta \mathbf{E},
\]

where \( \delta \mathbf{B} \) denotes the magnetic field strength averaged over the area of the Larmor circle, and \( \delta \mathbf{E} \) is the electric field averaged over its periphery.

We now specialize to perturbations corresponding to ideal MHD modes in a low-beta, high-aspect-ratio tokamak. The perturbation of the parallel magnetic field is then negligible, i.e., \( \delta B_\parallel = 0, \delta B_\perp = 0 \), as follows from the lowest-order minimization of the MHD energy functional. Since \( \kappa = \nabla \times \mathbf{B} = - \nabla R \), with \( R \) the major radius, and the ideal MHD displacement \( \delta \mathbf{r} \) is related to the electric field by

\[
\delta \mathbf{E} = i \omega \mathbf{\xi} \times \mathbf{B},
\]

the gyro-averaged of \( Z e \mathbf{v} \cdot \delta \mathbf{E} \) becomes

\[
-i \omega M (v^2 / 2 + v^2) \mathbf{\xi} \cdot \mathbf{\kappa}.
\]

We now obtain the gyro-averaged perturbation of the distribution function as

\[
\delta f = i \omega (\omega - n \omega_\parallel) \frac{\partial f}{\partial \mathcal{E}} \int_{-\infty}^{t} M \left( \frac{v^2}{2 + v_\parallel^2} \right) \mathbf{\xi} \cdot \mathbf{\kappa} \, dt',
\]

where

\[
\omega_\parallel = \frac{\partial f / \partial p_\varphi}{\partial f / \partial \mathcal{E}}.
\]

The first term in \( \delta f \) is the kinetic response and the second term the fluid response of the fast particles to the perturbation. The energy associated with these perturbations is

\[
\frac{\delta W_k}{\delta W_f} = \frac{1}{2} \int \xi_\parallel \cdot \nabla \left( \frac{\delta \mathbf{P}_k}{\delta \mathbf{P}_f} \right) d^3 r,
\]

where the integration is taken over the entire plasma, and \( \delta \mathbf{P}_k = \delta \mathbf{p}_{\perp k} + (\delta \mathbf{p}_{\perp k} - \delta \mathbf{p}_{\parallel k}) \mathbf{b} \) is the pressure tensor with \( \delta \mathbf{p}_{\perp k} \) and \( \delta \mathbf{p}_{\parallel k} \) the perpendicular and parallel pressure perturbations associated with \( \delta \mathbf{r}_k \), respectively. A similar definition applies to \( \delta \mathbf{P}_f \). The reader is referred to Refs. 12 and

![Fig. 1. A fast ion Larmor orbit, C, consisting of circular gyromotion superimposed on a slow drift \( \Delta \mathbf{R} \).](image-url)
vanishing magnetic shear, i.e., to the region where the mode studied in Ref. 6 is localized to the region of nearly the orbit average

\[ L(t) = \int_{t-	au_b}^{t} \langle \mathcal{E} \rangle dt. \]

To recast this result in a convenient form, we introduce the orbit average

\[ \langle \cdots \rangle = \frac{1}{\tau_b} \int_{0}^{\tau_b} \cdots dt, \]

where \( \tau_b = 2\pi/\omega_\ast \) is the time required to complete a poloidal orbit, i.e., \( \tau_b \) is the bounce time for trapped particles and the transit time for circulating ones. Since the function \( L = -Mv^2/2 + v^2 \) varies as \( e^{-i(\omega + n\phi)} \) and is otherwise a \( \tau_b \)-periodic function of time, it can be written as a Fourier series,

\[ L(t) = e^{-i(\omega + n\phi)} \sum_{l=-\infty}^{\infty} \langle \mathcal{E} e^{i(\omega - l\omega_b + n\phi)} \rangle e^{i(l\omega_b)}. \]

Finally, using Refs. 12, 17,

\[ d^3r \int d^3v = \frac{2\pi^2 v}{ZeB_0} \int_{\Lambda} -\Lambda \int d\phi \int d\phi \int d\phi, \quad (2) \]

where \( \tau \) is the time along the orbit, \( \sigma = v_1/|v_1| \), and \( \Lambda = \mu B_0/\Omega \) with \( B_0 \) the magnetic field strength on the magnetic axis, we can write the kinetic perturbation of the fast-ion energy as

\[ \delta W_k = -\frac{\pi^2 M^2}{ZeB_0} \sum_{\sigma} \int \langle v^2 \rangle \int d\phi \int d\Lambda \tau_b \frac{df}{d\phi} \]

\[ \times \sum_{l} \langle \mathcal{E} e^{i(\omega - l\omega_b + n\phi)} \rangle \left| \frac{v^2}{2} + v^2 \right| \xi_{1l} \]

\[ \cdot \kappa_{\sigma} e^{i(\omega - l\omega_b + n\phi)} \rangle \right| ^2, \quad (3) \]

which generalizes an almost identical expression in Ref. 12 to an arbitrary Larmor radius. Our remaining task is to evaluate this multiple integral for the case at hand.

The dominant \((m,n)\) harmonic of the “double kink” mode studied in Ref. 6 is localized to the region of nearly vanishing magnetic shear, i.e., to the region where \( q = q_{\text{min}} = m/n \). In a plasma with the circular poloidal cross section the leading-order displacement is of the form

\[ \xi_{1l}(r) = \xi_{0l}(r) e^{i(m\theta - n\phi - \omega t)}, \]

where \( \theta \) is the poloidal angle. From the lowest-order minimization of the MHD energy functional it follows that

\[ \frac{d^2r}{d^2r} + \im \xi_{0\theta} = 0 \]

(assuming incompressibility) and since \( \kappa = -\left( \mathbf{r} \cos \theta - \mathbf{\tau} \sin \theta \right)/R \), we have

\[ \bar{\xi}_{0l} \cdot \kappa = -\left( \bar{\xi}_{0l}/R \right) (\cos \theta + \cdots) e^{i(m\theta - n\phi - \omega t)}, \]

where the omitted terms are odd in \( \theta \). The mode frequency is expected to be smaller than the bounce frequency of fast ions, \( \omega \ll \omega_b \), which allows us to recast (3) in the form

\[ \delta W_k = -\frac{\pi^2 M}{\Omega} \sum_{\sigma} \int \langle v^2 \rangle \int d\phi \int d\Lambda \\
\times \int \frac{df}{d\phi} \frac{d\phi}{d\omega} \left( \frac{v^2}{2} + v^2 \right) \frac{\xi_{0l}}{R} \cos \theta \right|^2. \]

(4)

Here, we have assumed that the trajectories approximately follow magnetic field lines, i.e., we have taken the orbit width to be smaller than the minor radius, \( \Delta \ll r \), so that the helical angle \( m\theta - n\phi \) remains approximately constant along the orbit. Then, for trapped particles, \( L(t) = e^{-i(\omega + n\phi)} \sum_{l=-\infty}^{\infty} \langle \mathcal{E} e^{i(\omega - l\omega_b + n\phi)} \rangle e^{i(l\omega_b)}, \]

\[ \langle \psi \rangle = \frac{v^2}{2} \Omega \mu R^2 \],

where \( \Omega = ZeB_0/M \) and we have taken the magnetic shear to be weak; for untrapped particles \( \langle \psi \rangle = v_1/R \approx \omega_b/q \).

In the same approximation that underlies (4), the energy associated with the fluid response becomes

\[ \delta W_f = \frac{1}{2R} \int \bar{\xi}_{0l} \cdot \kappa \frac{d(p_\perp + p_\parallel)}{d\phi} \cos \theta d^3r \]

\[ = \frac{\pi^2 M}{2qR} \sum_{\sigma} \int \xi_{0l} \frac{d\Lambda}{d\phi} dr \int v^2 \frac{dv}{d\phi} \frac{\xi_{0l}}{R} \cos \phi d\phi, \]

(5)

where we have used (2), and the perpendicular and parallel pressures are defined by

\[ \left( \frac{p_\perp}{p_\parallel} \right) = \int M \left( \frac{v^2}{2} \right) f(v) d^3v. \]

### III. SMALL MODE FREQUENCY

In this section, we study the response of the fast ions in the adiabatic limit \( \omega \rightarrow 0 \). More specifically, we take the mode frequency to be smaller than the toroidal precession frequency, \( \omega \ll \omega_\ast \). Since

\[ \omega \sim \frac{Mv^2}{ZeRB_0} \sim e^{i(\phi)}, \]

we therefore also have \( \omega \ll \omega_\ast \). The integral (4) over the trapped region in velocity space may in this case be simplified to

\[ \delta W_k = -\frac{\pi^2 M}{2qR} \int r dr \int d\Lambda \int \frac{df}{d\phi} \frac{\xi_{0l}}{R} \cos \phi \right|^2 v^2 dv. \]

(6)
A. Small orbit width

If the orbit is narrow in comparison with the mode, \( \Delta_b \ll \Delta_m \), the amplitude \( \bar{\xi}_0 \) is approximately constant along the orbit, which simplifies the integral (6). Using Ref. 18,

\[
\tau_b(\cos \theta) = \frac{8qR}{v \sqrt{2} \epsilon} [2E(k) - K(k)],
\]

where \( \epsilon = r/R < 1 \),

\[
k^2 = \frac{1 + \epsilon - \Lambda}{2\epsilon},
\]

is the trapping parameter with \( 0 < \epsilon < 1 \), and \( K(k) \) and \( E(k) \) are complete elliptic integrals, we can rewrite (6) as

\[
\delta W'^{rr} = \delta W'^{rr} = -2^{3/2} \pi^2 MR \int_0^\infty \frac{dv}{v^4} \int_0^{a/\xi_0} e^{2v^2/2} \times dr \int_0^1 \frac{df}{dr} [2E(k) - K(k)] d \bar{k}^2,
\]

with \( a \) the minor radius of the plasma edge. This expression is identical to the fast-ion energy perturbation for the ordinary \( m = n = 1 \) internal kink mode.\(^{19}\) In particular, for an isotropic fast-ion distribution \( (p_\perp = p_\| = p) \), the fast-ion action is stabilizing and we have

\[
\delta W'^{rr} = -2^{3/2} \frac{\pi^2}{R \xi_0} \int_0^{a/\xi_0} \frac{dp}{d \bar{r}} r^{3/2} dr,
\]

where we have used \( \int_0^1 [2E(k) - K(k)] d \bar{k}^2 = 2/3 \).

B. Large orbit width

We now consider the limit in which the mode width is much smaller than the banana width, which is still assumed to be smaller than the minor radius, \( \Delta_m \ll \Delta_b \ll r \).

The orbit average \( \langle \bar{\xi}_0 \cos \theta \rangle \) appearing in (6) then reduces to

\[
\langle \bar{\xi}_0 \cos \theta \rangle = \frac{1}{\tau_b} \int \bar{\xi}_0 \cos \theta \frac{d \bar{r}}{dr} = -\frac{2X}{v_d \tau_b} \cot \theta_a,
\]

where

\[
X = \int_0^{a/\xi_0} \bar{\xi}_0 \ d \bar{r}
\]

measures the total mode strength, and \( \theta_a \) is the poloidal angle \( \theta \) at which the orbit intersects the mode, see Fig. 2. Since

\[
v_d = \frac{v^2 / 2 + v_\|^2}{\Omega R},
\]

we can now write (6) as

\[
\delta W'^{rr} = -2^{3/2} \frac{\pi^2}{e \Delta_0} \frac{MR \Omega^2 X^2}{v_d} \int \frac{d \bar{r}}{dr} \cot^2 \theta_a \frac{d \Lambda}{dr} 2E(k) - K(k),
\]

where \( r \) refers to the radial position of the banana tip, and all quantities in front of the integral should be evaluated at the radius of the mode. The integration is carried out over all orbits that intersect the mode. To perform the \( \Lambda \)-integral, we note that the excursion of a trapped orbit from its average flux surface \( r \) is

\[
\Delta r(\theta) = \frac{v}{\Omega_\theta} \sqrt{1 - (1 - \epsilon \cos \theta)},
\]

so that \( \theta_a \) is determined by

\[
\Lambda(1 - \epsilon \cos \theta_a) = 1 - \left( \frac{\Omega_\theta (r_m - r)}{v} \right)^2,
\]

where \( r_m \) is the location of the mode. Hence \( d \Lambda = -\epsilon \sin \theta_a d \theta_a \), and the \( \Lambda \)-integral may be written as

\[
\int \frac{d \bar{r}}{dr} \frac{\cot^2 \theta_a d \Lambda}{2E(k) - K(k)} \approx \int \frac{d \bar{r}}{dr} \frac{\epsilon d \theta_a}{2E(k) - K(k)}
\]

The \( \theta_a \)-integral is dominated by the contribution from particles with small \( \theta_a \) whose orbits are tangential to the mode near the outer midplane, reflecting the fact that these particles spend the longest time in the mode. Formally, the integral is logarithmically divergent for \( \theta_a \rightarrow 0 \), but it is cut off at some minimum angle \( \theta_a \sim 1/v \), below which (10) does not hold. This happens when \( \theta_a \) becomes so small that the two points \( \theta = \pm \theta_a \) where the orbit intersects the mode become indistinguishable., i.e., when the section of the orbit that connects these two points lies entirely within the mode. The condition for this is that

\[
\Delta r(\theta = 0) - \Delta r(\theta = \theta_a) < \Delta_m,
\]

where

\[
\frac{\Delta r(\theta_a)}{\Delta r(0)} \approx 1 - \frac{\theta_a^2}{8k^2},
\]

from (7) and (12). Hence the cut-off angle is of the order of \( \theta_a \sim \sqrt{\Delta_m / \Delta_b} \). Finally, recalling that the orbit average is smeared out over the length scale of the Larmor radius, we conclude that the logarithmic cut-off is roughly

\[
\ln \theta \sim \ln \left( \frac{\Delta_b}{\max(\rho, \Delta_m)} \right).
\]

FIG. 2. A wide banana orbit intersecting the narrow mode structure at the poloidal angle \( \theta = \theta_a \).
Formally, this is a large parameter; in practice it is not very large and quite insensitive to the argument under the square root.

To proceed with the evaluation of (11), we write this integral as

$$\delta W'_h = -4 \pi^2 e^2 q^{-1} M \Omega R X^2 \ln \theta \int_0^\infty v^3 dv \times \int_0^1 \frac{\partial f}{\partial r} \frac{dK}{2E(k) - K(k)},$$

where, again, all quantities are to be evaluated at the mode radius and we have replaced the integration variable $r$ by $k$ by making use of (12). If the fast ions are isotropic, the distribution function can be pulled out of this integral, which can then be evaluated numerically, and we obtain

$$Re\delta W'_h = -32.6 e Z e B f R X^2 \ln \theta \int_0^\infty \frac{\partial f}{\partial v} v^3 dv.$$  \hspace{1cm} (13)

Again, the fast ions are seen to be stabilizing. It is interesting to note that the ratio between the narrow (8) and wide (13) orbit limits is

$$\frac{\delta W'_h(\Delta_b < \Delta_m)}{\delta W'_h(\Delta_m < \Delta_b)} \sim \frac{\Delta_e}{\Delta_m \ln \theta},$$  \hspace{1cm} (14)

for a slowing-down distribution $f \propto v^{-3}$. If the banana width and the mode width are comparable, which is likely in practice, the two limiting expressions (8) and (13) give fairly similar results. If the banana width exceeds the mode width, the fast ions spend only a fraction of their time in the region of the mode and the customary small-banana-width approximation leads to an overestimate of their stabilizing action, as previously noted for conventional sawteeth. \cite{20}

**IV. FINITE FREQUENCY**

**A. Small orbit width**

We now turn to the case when the frequency $\omega$ is comparable to the toroidal precession frequency ($\bar{\omega}$). First, we consider the limit of thin orbits, $\Delta_b \ll \Delta_m$. As might be expected from the analysis in Sec. III A, the situation is again entirely equivalent to that of the usual $m=n=1$ kink.\cite{9,20} Neglecting the term $\omega \ll \omega_+$ in the numerator of (4),\cite{21} and adding the remainder to the fluid part (5) of the energy perturbation gives the following expression for the total energy integral for trapped hot ions:

$$\delta W''_h = \delta W'_h + \delta W' k = \frac{\pi^2 M v_{\text{max}}^6}{4R} \int_0^a \frac{\xi e q^{-1}}{\xi_0} r dr \int_0^1 x^2 dx \times \int_0^1 \frac{\partial f}{\partial r} \tau_0 (\cos \theta) \frac{\omega}{\omega - x} d\Lambda,$$ \hspace{1cm} (15)

where $x = (v/v_{\text{max}})^2$ is the velocity normalized to some maximum velocity $v_{\text{max}}$, and $\tau_0 = \omega / n(\phi) v - v_{\text{max}}$. We shall consider the case when the fast ions are produced by neutral-beam injection, and choose the velocity $v_{\text{max}}$ to be equal to the injection velocity.

For simplicity, following Refs. 9 and 20, we assume that beam ions are injected perpendicular to the magnetic field with a slowing-down speed distribution,

$$f(r,x,s) = N(s) \delta (\Lambda - 1) x^{-3/2},$$  \hspace{1cm} (16)

where the constant $N(s)$ is related to the flux surface averaged pressure by

$$\langle p \rangle = \frac{\Gamma^2(1/4)}{6 \sqrt{2} \pi \epsilon} N M v_{\text{max}}^5.$$  

An expression similar to (16), with $x^{-3/2}$ replaced by an exponential factor, could be used to represent the velocity distribution of ions heated by waves in the ion cyclotron range of frequencies (ICRF). White and co-workers\cite{20} have found that the stability properties of the finite frequency $m=1$ internal kink mode are not strongly dependent on the exact form of the energetic particle distribution: it is reasonable to assume that this is also true of the finite frequency double kink mode considered here.

Substituting the distribution function (16) in (15) and taking $\Delta_m \ll r$ gives

$$\delta W''_h = C \omega \ln (1 - \omega^2),$$ \hspace{1cm} (17)

with

$$C = -\pi^2 (2R)^{1/2} \left[2E(2^{-1/2}) - K(2^{-1/2})\right] M v_{\text{max}}^5 \times \int \xi_0^2 N(r)^{1/2} dr.$$  

The dispersion relation (1) thus becomes

$$-i\omega + \delta W_c + \dot{C} \omega \ln (1 - \omega^2) = 0,$$ \hspace{1cm} (18)

where we have introduced the normalized quantities

$$\dot{\omega}_A = \frac{m q \omega A}{\sqrt{1 + 2q^2(|s_1| + |s_2|)}},$$

$$\delta W_c = \frac{\mu_0 R}{2 \pi r^2 \xi_0 B^2} \delta W_c,$$

$$\dot{C} = \frac{\mu_0 R}{2 \pi r^2 \xi_0 B^2} C.$$

The logarithm in (18) has an imaginary term equal to $\pi i$ if $\omega < 1$ and if $1/2 < \omega < 1$, the real part is negative and thus destabilizing. The necessary ingredients for a “fishbone” instability below the ideal MHD threshold are thus present.

Moreover, by inspecting (15) it is clear that this conclusion should be largely insensitive to the specific choice of distribution function $f$. In general, the resonant denominator gives rise to an imaginary part in $\delta W_h$ and the real part is expected to be negative in the frequency range $\omega \sim 1$ under fairly general conditions. In the case of the distribution function (16), the instability threshold in hot-ion beta, $\beta_h = 2\mu_0 B / B^2$, is obtained from the imaginary part of (18),

$$\frac{\omega}{\omega_A} = \dot{C} (\pi \dot{\omega}_A + \gamma \ln |1 - \omega^{-1}|),$$

where $\omega = \omega_r + i \gamma$ (with $\gamma \ll \omega_r$), which may be written as

$$\gamma \dot{C} \ln |1 - \omega^{-1}| = \frac{\omega_r}{\omega_A} \left(1 - \frac{\beta_h}{\beta_{\text{crit}}}\right),$$

where $\beta_{\text{crit}}$ is the instability threshold in hot-ion beta. \cite{20}
with \( \beta_{\text{crit}} = -n(\hat{\phi})_{v=v_{\text{max}}} / \pi \hat{C} \omega_A \). This shows that instability may occur if

\[
\beta_h > \beta_{\text{crit}} = -\frac{n(\hat{\phi})_{v=v_{\text{max}}}}{\omega_A},
\]

since \( \hat{C} \) is of the order \( \beta_h / \epsilon \).

If bulk ion diamagnetic effects are retained, a second branch of the dispersion relation appears, containing low-frequency fishbones.\textsuperscript{10, 11, 15}

**B. Large orbit width**

If the mode frequency is finite and the hot-ion orbits are wide, we use the ordering (9) and the orbit average (10) to write the kinetic part of the energy perturbation of the trapped fast ions (4) as

\[
\delta W_{k}^r = 4 \pi^2 n M \Omega X^2 \int v^3 \, dv \int d\tau \frac{\partial f}{\partial \tau} \tau_n(\omega + n(\hat{\phi})) \, d\Lambda.
\]

This is the finite-frequency analogue of (11), and, as in (15), we have neglected the term \( \omega \) in the numerator of (4). Cutting off the divergent \( \Lambda \)-integral as in Sec. III B, we obtain, after some algebra,

\[
\delta W_{k}^r = 2 \pi^2 \epsilon^2 q^{-1} M \Omega X^2 v_{\text{max}}^4 \ln \int_0^1 x^2 \, dx \times \int_0^1 \frac{df}{d\tau} \, dk \frac{d}{d \tau} \left( 2E(k) - K(k) \right) (\omega - x),
\]

where the dimensionless variables \( \omega \) and \( x \) are defined after (15). Finally, the use of the distribution function (16) modelling perpendicular beam injection gives

\[
\delta W_{k}^r = M \Omega X^2 v_{\text{max}}^4 N'(r) \times \frac{2 \epsilon^2 q^{-1} \ln \theta}{2E(2-\epsilon^2) - K(2-\epsilon^2)} \left( \omega^{1/2} \ln \left( 1 + \omega - \omega^{-1/2} \right) - \omega^{-1/2} \right),
\]

for the kinetic fast-ion response, and

\[
\delta W_{f}^r = \pi^2 \sqrt{2 \epsilon} m v_{\text{max}}^5 N'(r) R \int_0^2 \xi_0^2 \, d\tau,
\]

for the fluid response. As in the case of narrow orbits, \( \delta W_{f}^r \) has an imaginary part equal to \( \pi i \) if \( \omega < 1 \), and a destabilizing real part. The structurally necessary components for a fishbone instability are thus again present. Comparing the driving force of the instability in the narrow (17) and wide (19) orbit limits gives the same estimate (14) as obtained above for the stabilizing effects of the hot ions on the low-frequency (MHD) kink mode.

**V. CONCLUSIONS**

As shown in Ref. 6, the plasma in a large-aspect-ratio tokamak with negative magnetic shear is unstable to ideal MHD “double kink” modes if the plasma beta is sufficiently high. The calculation carried out in the present paper shows that the stability threshold is influenced by the presence of hot ions, produced e.g., by fusion reactions or neutral-beam injection. The ideal MHD mode is stabilized above this threshold by the same mechanism thought to be responsible for sawtooth stabilization. Furthermore, if the hot-ion beta exceeds a critical value, a new instability appears below the MHD threshold; this mode is analogous to the fishbone instability associated with the usual \( m = n = 1 \) internal kink mode. In short, fast ions are expected to interact with double kinks and ordinary kinks in the same way. Orbit width effects are probably more important for double kink modes, and are shown to change the strength of the interaction somewhat. In the case of narrow orbits the interaction energy is proportional to the mode width; in the case of wide orbits it is proportional to the square of the mode width.

We close by commenting on the role of bulk ion diamagnetic effects, i.e., a finite plasma diamagnetic frequency. These effects have been neglected here as we have used the ideal MHD description of the bulk plasma, but are important when the fishbone mode frequency is of the same order as the ion diamagnetic frequency of the background plasma. When retained, they introduce a second type of fishbone\textsuperscript{10, 11, 15} excited by the fast ions. Since we have seen that the underlying physics is the same whether the fast ions interact with ordinary or double kinks, such fishbones are possible also in the present context. In particular, they can be excited not only by trapped ions, but also by circulating ones as noted by Betti and Freidberg.\textsuperscript{22}

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13Strictly speaking, this term is not very small. It appears to have been neglected in Ref. 9 and later retained in Ref. 20, where it was shown not to influence the final result significantly.