Axisymmetric two-fluid plasma equilibria with momentum sources and sinks

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Abstract

Axisymmetric plasma equilibria with toroidal flow are investigated using a two-fluid analysis, taking into account the presence of momentum and current drive, together with dissipative effects. The equilibria are described by a set of three equations describing the spatial variation of poloidal magnetic flux, plasma pressure and toroidal magnetic field. These equations indicate that magnetic flux surfaces do not in general rotate as rigid bodies, as required by ideal magnetohydrodynamics (MHD) in the absence of momentum sources and poloidal flows. For specific momentum drive and damping scenarios, expressions are obtained giving the variation of density on flux surfaces and a relation between temperature and toroidal rotation rate that can in principle be tested experimentally. A simple relation between the loop voltage and plasma current in an inductive tokamak plasma with toroidal flow is also derived.

1. Introduction

The original Grad-Shafranov equation, describing axisymmetric magnetohydrodynamic (MHD) plasma equilibria without flow [1, 2], has been generalised over the past half century to include the effects of both toroidal and poloidal flow [3, 4], and also multi-fluid effects [5, 6, 7]. These analyses are generally based on one or more momentum balance equations which do not take into account the presence of momentum sources or sinks, i.e. torques or dissipation. Tokamak plasmas can have substantial toroidal flows, with rotation velocities approaching [8] or even exceeding [9] the local sound speed, but the presence of dissipative processes (in particular neoclassical and turbulent viscosity) means that these flows must be continuously driven, usually through the injection of neutral particle beams which are ionized in the plasma and subsequently transfer their momentum to it. A complete theoretical analysis of tokamak plasma equilibria with flow should therefore take into account the presence of momentum drive and dissipation. This analysis, when applied to measurements of plasma and field profiles, could in principle be used to obtain experimental information on momentum transport. A second motivation for such a study is that it can provide a recipe for evolving realistically the plasma equilibrium in global toroidal turbulence codes, such as the two-fluid `CENTORI` code [10].

In section 2 we define a set of non-orthogonal plasma-based coordinates that are well-suited for the description of axisymmetric plasma equilibria with sources and sinks of momentum and current. The steady-state momentum balance equations for the ion and electron fluids are written down in these coordinates in section 3, assuming purely toroidal flows, and are used to derive expressions for the variation of density on flux surfaces and a relation between toroidal rotation rate and temperature, which can in principle be tested experimentally. Finally in this section we derive an Ohm's law for inductive tokamak plasmas with driven flows, i.e. a relation between the loop voltage and the plasma current. Our conclusions appear in section 4.

2. Plasma coordinates

Axisymmetric magnetic fields can always be written in terms of right handed cylindrical coordinates (R, φ, Z) in the form

$$\mathbf{B} = \nabla\Psi \times \nabla\varphi + RB_\varphi \nabla\varphi, \tag{1}$$

where Ψ (poloidal magnetic flux) and B_φ (toroidal magnetic field) are functions of major radius R, vertical distance Z and possibly time but not toroidal angle φ. Rather than (R, φ, Z), it is often convenient to use a set of right-handed plasma-based coordinates (Ψ, ϑ, ζ) where toroidal angle $\zeta = -\varphi$ and poloidal angle ϑ may be defined in a number of ways. One may for example choose ϑ such that $\nabla\vartheta$ is parallel to $\nabla\Psi \times \nabla\varphi$, so that $(\nabla\Psi, \nabla\vartheta, \nabla\zeta)$ is an orthogonal set of basis vectors. However it is not essential to define ϑ in this way; it is sufficient that the following Jacobian does not generally vanish in the spatial domain of interest:

$$J \equiv (\nabla\Psi \times \nabla\vartheta) \cdot \nabla\zeta = \frac{1}{R}\frac{\partial\,(\Psi, \vartheta)}{\partial\,(R, Z)} = \frac{|\nabla\Psi|}{R}\frac{\partial\vartheta}{\partial l}. \tag{2}$$

Here l denotes arc length along a flux surface in the (R, Z) plane. In the present paper we choose the Jacobian to be a flux function, but we do not require it to be a constant, as in the case of standard Hamada coordinates [11]. This choice facilitates the evaluation of flux-surface averages, since the volume element $Rdld\zeta d\Psi/|\nabla\Psi|$ is then equal to $d\vartheta d\zeta d\Psi/J$ and, since $J = J(\Psi)$, integrals over ϑ and ζ do not depend on this quantity. In general such a coordinate system is not orthogonal but quasi-orthogonal, meaning that two of the basis vectors (in this case $\nabla\Psi$ and $\nabla\vartheta$) have a non-vanishing scalar product. When J is a flux function ϑ may be determined by first solving (numerically, if necessary) a pair of Hamiltonian ordinary differential equations describing curves of constant Ψ in the (R, Z) plane:

$$\frac{dR}{dl} = -\frac{1}{|\nabla\Psi|}\frac{\partial\Psi}{\partial Z}, \tag{3}$$

$$\frac{dZ}{dl} = \frac{1}{|\nabla\Psi|}\frac{\partial\Psi}{\partial R}. \tag{4}$$

These equations can be used to compute $R(\Psi, l)$ and $Z(\Psi, l)$. Simultaneously one may compute the integral

$$\int_0^l \frac{R(\Psi, l')}{|\nabla\Psi|} dl' = \frac{\vartheta}{J(\Psi)}. \tag{5}$$

The value of J for the flux surface in question is then fixed by requiring that ϑ be equal to 2π when R and Z have returned to their initial values. Expressing l as a function of ϑ using equation (5), we can finally obtain R and Z as functions of Ψ and ϑ.

In terms of ζ the full magnetic field is given by an expression of the form

$$\mathbf{B} = \nabla\zeta \times \nabla\Psi + RB_\zeta\nabla\zeta, \tag{6}$$

where $B_\zeta = -B_\varphi$. We denote RB_ζ by F. In the absence of poloidal flows, momentum sources and dissipation, steady state toroidal momentum balance requires that F be a flux function in the ideal MHD limit [5]. This is also true in two-fluid theory when electron inertial effects are neglected [6]. However since in the present paper we are allowing for the possibility of toroidal momentum drive and damping, we will assume only that F is axisymmetric, i.e. $F = F(R, Z)$ or, equivalently, $F = F(\Psi, \vartheta)$. Using Ampère's law it is straightforward to show that the current density can be written in the form

$$\mathbf{j} = Rj_\zeta\nabla\zeta - \frac{1}{\mu_0}\frac{\partial F}{\partial\Psi}\nabla\zeta \times \nabla\Psi - \frac{1}{\mu_0}\frac{\partial F}{\partial\vartheta}\nabla\zeta \times \nabla\vartheta, \tag{7}$$

where μ_0 is the permeability of free space and the toroidal current density j_ζ is given by the expression

$$j_\zeta = \frac{1}{\mu_0 R}\left\{R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) + \frac{\partial^2\Psi}{\partial Z^2}\right\} \equiv \frac{1}{\mu_0 R}\triangle^*\Psi. \tag{8}$$

Using equations (2) and (6-8) we deduce that the Lorentz force $\mathbf{j} \times \mathbf{B}$ can be written as

$$\mathbf{j} \times \mathbf{B} = -\frac{1}{\mu_0 R^2}\left\{\triangle^*\Psi + F\frac{\partial F}{\partial\Psi}\right\}\nabla\Psi - \frac{F}{\mu_0 R^2}\frac{\partial F}{\partial\vartheta}\nabla\vartheta + \frac{JF}{\mu_0}\frac{\partial F}{\partial\vartheta}\nabla\zeta. \tag{9}$$

In the next section we will use the expressions obtained above in the equilibrium ion and electron momentum balance equations.

3. Two-fluid momentum balance

We consider a quasi-neutral two-fluid plasma consisting of singly-charged ions and electrons, each with a scalar pressure, and assume that the ion fluid has a purely toroidal flow $\mathbf{v}_i = R^2\Omega_\zeta\nabla\zeta$. It is straightforward to show that the ion momentum balance equation can then be written in the form

$$-\frac{1}{2}m_i n\Omega_\zeta^2\nabla R^2 = -\nabla p_i - ne\nabla\Phi - ne\Omega_\zeta\nabla\Psi + \mathbf{F}^{\rm ext} + \mathbf{F}^{\rm drag} + n\mathbf{R}_{ie}, \qquad (10)$$

where m_i is ion mass, $-e$ is electron charge, n is the common electron and ion number density, p_i is ion pressure, Φ is electrostatic potential, $\mathbf{F}^{\rm ext}$ is the applied external force per unit volume (for example due to neutral beam injection), $\mathbf{F}^{\rm drag}$ is the rate of momentum loss per unit volume (for example due to neoclassical or turbulent viscosity), and $\mathbf{R}_{ie}$ is the rate at which a single ion exchanges momentum with electrons. In the case of inductive tokamak operation we can write

$$\mathbf{R}_{ie} = \frac{eV_{\rm L}}{2\pi}\nabla\zeta - e\eta\mathbf{j}, \qquad (11)$$

where $V_{\rm L}$ is the loop voltage and η is resistivity (assumed to be isotropic).

The corresponding electron momentum balance equation (effectively the generalized Ohm's law for this problem) can be written in the form

$$\mathbf{0} = -\nabla p_e + ne\nabla\Phi + ne\Omega_\zeta\nabla\Psi + \mathbf{j}\times\mathbf{B} + n\mathbf{R}_{ei}, \qquad (12)$$

where p_e is electron pressure and $\mathbf{R}_{ei}$ is the rate at which a single electron exchanges momentum with ions; momentum conservation requires that

$$\mathbf{R}_{ei} = -\mathbf{R}_{ie} = -\frac{eV_{\rm L}}{2\pi}\nabla\zeta + e\eta\mathbf{j}. \qquad (13)$$

In equation (12) we have neglected inertial terms, together with momentum sources and sinks. The former assumption is justified by the fact that electron bulk flows in tokamak plasmas are invariably negligible compared to the electron thermal speed. The neglect of momentum sources and sinks in equation (12) is justified by the fact that any momentum acquired by an electron via its interaction with, for example, beam ions is very rapidly transferred to bulk ions (if this were not the case, neutral beam injection would produce large numbers of highly superthermal electrons, which are not observed, except during tokamak disruptions or lower hybrid current drive). Thus, although one could include $\mathbf{F}^{\rm ext}$ and $\mathbf{F}^{\rm drag}$ terms in equation (13), these terms must cancel almost exactly, and consequently they can be neglected. In principle one should also include external current sources and the bootstrap current, together with neoclassical resistivity and thermal force terms. These additional effects will, in general, modify the relation between the loop voltage and the plasma current, and can be included in more detailed descriptions relating to specific experimental situations where they are expected to play significant roles.

Adding equations (10) and (12), and using equation (13), we obtain

$$-\frac{1}{2}m_i n\Omega_\zeta^2\nabla R^2 = -\nabla p + \mathbf{j}\times\mathbf{B} + \mathbf{F}^{\rm ext} + \mathbf{F}^{\rm drag}, \qquad (14)$$

where $p = p_i + p_e$. To ensure compatibility with the assumption of zero poloidal flow, we consider only the toroidal components of both the external momentum drive $\mathbf{F}^{\text{ext}}$ and the momentum loss rate, $\mathbf{F}^{\text{drag}}$ (poloidal flows in tokamak plasmas are subject to very strong damping due to neoclassical effects [12]; thus, to a good approximation we can assume that the poloidal components of $\mathbf{F}^{\text{ext}}$ and $\mathbf{F}^{\text{drag}}$ cancel. Turbulence-driven poloidal flows exceeding neoclassical levels can occur in tokamak plasmas, although these generally have Mach numbers that are significantly lower than those of toroidal flows [13]. If sheared, such poloidal flows can influence transport but do not change the basic equilibrium). For simplicity we will assume moreover that the toroidal momentum loss rate can be characterised by a phenomenological relaxation time, τ_ζ, i.e.

$$\mathbf{F}^{\text{drag}} = -\frac{m_i n \Omega_\zeta R^2}{\tau_\zeta} \nabla\zeta. \tag{15}$$

In the case of momentum losses arising from neoclassical or turbulent viscosity, a more exact expression for $\mathbf{F}^{\text{drag}}$ would involve spatial derivatives of the toroidal ion velocity. However the approximation represented by equation (15) is sufficiently accurate for our purposes.

Denoting the toroidal momentum drive by F_ζ^{ext} and using the expression for the Lorentz force given by equation (9), we find that equation (14) can be written in the form

$$\frac{1}{\mu_0 R^2}\left\{\triangle^*\Psi + F\frac{\partial F}{\partial \Psi}\right\}\nabla\Psi = -\frac{F}{\mu_0 R^2}\frac{\partial F}{\partial \vartheta}\nabla\theta + \frac{J}{\mu_0}\frac{\partial F}{\partial \vartheta}\nabla\zeta$$

$$+\frac{1}{2}m_i n \Omega_\zeta^2 \nabla R^2 - \nabla p + \left[F_\zeta^{\text{ext}} R - \frac{m_i n \Omega_\zeta R^2}{\tau_\zeta}\right]\nabla\zeta. \tag{16}$$

Here R^2 and p may be regarded as functions of Ψ and ϑ, and hence we can write

$$\nabla R^2 = \frac{\partial R^2}{\partial \Psi}\nabla\Psi + \frac{\partial R^2}{\partial \vartheta}\nabla\vartheta, \tag{17}$$

$$\nabla p = \frac{\partial p}{\partial \Psi}\nabla\Psi + \frac{\partial p}{\partial \vartheta}\nabla\vartheta. \tag{18}$$

Using these relations in equation (16), and equating the $\nabla\Psi$, $\nabla\vartheta$ and $\nabla\varphi$ components of this equation, we obtain

$$\frac{1}{\mu_0 R^2}\left\{\triangle^*\Psi + F\frac{\partial F}{\partial \Psi}\right\} = -\frac{\partial p}{\partial \Psi} + \frac{1}{2}m_i n \Omega_\zeta^2 \frac{\partial R^2}{\partial \Psi}, \tag{19}$$

$$\frac{\partial p}{\partial \vartheta} = \frac{1}{2}m_i n \Omega_\zeta^2 \frac{\partial R^2}{\partial \vartheta} - \frac{1}{\mu_0 R^2}F\frac{\partial F}{\partial \vartheta}, \tag{20}$$

$$F_\zeta^{\text{ext}} R + \frac{J}{\mu_0}\frac{\partial F}{\partial \vartheta} = \frac{m_i n \Omega_\zeta R^2}{\tau_\zeta}. \tag{21}$$

In the limit $\Omega_\zeta \to 0$ equation (19) reduces to the familiar form of the Grad-Shafranov equation. Moreover when p is expressed as a function of Ψ and R^2 rather than Ψ and ϑ, and F is assumed to be a function of Ψ only (a valid assumption in the limit $F_\zeta^{\rm ext} \to 0$, $\tau_\zeta \to \infty$ or when the toroidal momentum drive and damping terms in equation (21) cancel exactly), we recover from equation (19) the Grad-Shafranov equation for purely toroidal flow given by equation (75) in [6]. If on the other hand the drive and damping terms in equation (21) are finite and do *not* cancel exactly, F is not a pure flux function.

We can obtain an approximate upper limit on $\partial F/\partial\vartheta$ by considering the relative magnitudes of the three terms in equation (21). Using equation (2) and $F = RB_\zeta$ we can write

$$\frac{J}{\mu_0}\frac{\partial F}{\partial\vartheta} = \frac{JF}{\mu_0}\frac{\partial \ln F}{\partial\vartheta} \sim \frac{B_\zeta^2}{q\mu_0}\frac{\partial \ln F}{\partial\vartheta}, \qquad (22)$$

where $q = \mathbf{B}\cdot\nabla\zeta/\mathbf{B}\cdot\nabla\vartheta \sim 1$ is the plasma safety factor. Clearly the momentum source term on the left hand side of equation (21) and the plasma response term on the right hand side are comparable in magnitude. Hence the term proportional to $\partial F/\partial\vartheta$ cannot be significantly larger than the term on the right hand side, and it follows from equation (22) that

$$\frac{\partial F}{\partial\vartheta} \lesssim \frac{1}{\Omega_\zeta\tau_\zeta}\frac{v_\zeta^2}{c_A^2}F \equiv k_* F, \qquad (23)$$

where $v_\zeta = \Omega_\zeta R$ is the toroidal velocity and $c_A = B_\zeta/(\mu_0 m_i n)^{1/2}$ is the toroidal Alfvén speed. Although toroidal flows in tokamaks are invariably sub-Alfvénic ($v_\zeta < c_A$), toroidal rotation periods are generally much shorter than momentum confinement times ($\Omega_\zeta\tau_\zeta \gg 1$). Hence we infer that $k_* \ll 1$ and, to leading order in this parameter, F must be a flux function. The ϑ dependence of F can in principle be determined from equations (19-21) using a perturbation expansion in k_*.

3.1. Variation of density on flux surfaces

Some important consequences of equations (19-21) become apparent when one examines some simple, specific cases. We first consider a scenario in which F is a flux function and $F_\zeta^{\rm ext}\tau_\zeta = KR$, where K is a flux function. Equation (21) then implies that

$$m_i n\Omega_\zeta = K. \qquad (24)$$

Eliminating Ω_ζ from equation (20) and putting $p = n(T_e + T_i) \equiv 2nT$ where electron temperature T_e and ion temperature T_i are both assumed to be flux functions, we obtain

$$2T\frac{\partial n}{\partial\vartheta} = \frac{K^2}{2m_i n}\frac{\partial R^2}{\partial\theta}. \qquad (25)$$

This equation can be immediately integrated to give

$$n^2 = \frac{K^2}{2m_i T}R^2 + I(\Psi), \qquad (26)$$

where I is a flux function. Equivalently, we may write

$$n^2 = \langle n^2 \rangle + \frac{K^2}{2m_i T}\left(R^2 - \langle R^2 \rangle\right), \tag{27}$$

where angled brackets indicate flux surface averages. The variation of density on a flux surface indicated by this expression differs from the well-known result for rigidly-rotating flux surfaces in tokamak plasmas with purely toroidal flow when momentum sources and sinks are neglected [14]:

$$n = n_0(\Psi) \exp\left[\frac{m_i \Omega_\zeta^2 R^2}{4T}\right], \tag{28}$$

where in this case n_0 and Ω_ζ are both flux functions. Equation (28) can be obtained from equations (20) and (21) by assuming that F and $F_\zeta^{\rm ext}\tau_\zeta/nR$ are flux functions. Equations (27) and (28) have in common the property that n increases with R on a given flux surface, which arises from the presence of an inertial term in the ion momentum balance equation (10); this has been clearly observed in spherical tokamak plasmas with strong toroidal rotation [15, 16].

For the scenario considered above we can also obtain a relation between the measurable quantities n, Ω_ζ and T by eliminating K from equations (24) and (27):

$$n^2 = \frac{\langle n^2 \rangle}{1 + m_i \Omega_\zeta^2 \left(\langle R^2 \rangle - R^2\right)/2T}. \tag{29}$$

The consistency of this expression with equation (27) becomes apparent when one recognizes that Ω_ζ in this case is not a flux function.

Alternatively, we could consider a scenario in which F and $F_\zeta^{\rm ext}\tau_\zeta$ are flux functions. Denoting the latter by M, equation (21) indicates that $M = m_i n \Omega_\zeta R$ and equation (20) yields

$$2T\frac{\partial n}{\partial \vartheta} = \frac{M^2}{2m_i n R^2}\frac{\partial R^2}{\partial \vartheta}. \tag{30}$$

This equation can be integrated to give an expression of the form

$$n^2 = \frac{M^2}{2m_i T} \ln\left(\frac{R^2}{R_{\rm min}^2(\Psi)}\right) + n_{\rm min}^2(\Psi), \tag{31}$$

where $R_{\rm min}$ and $n_{\rm min}$ denote the minimum values of R and n on a particular flux surface. Eliminating M, we obtain an expression analogous to equation (29):

$$n^2 = \frac{n_{\rm min}^2}{1 - (m_i \Omega_\zeta^2 R^2/2T)\ln\left(R^2/R_{\rm min}^2\right)}. \tag{32}$$

As a final example, we consider a scenario in which F and $F_\zeta^{\rm ext}\tau_\zeta R$ are flux functions. It is evident from equation (21) that in this case the plasma angular momentum per unit volume $L = m_i n\Omega_\zeta R^2$ is a flux function and equation (20) becomes

$$2T\frac{\partial n}{\partial\vartheta} = \frac{L^2}{2m_i nR^4}\frac{\partial R^2}{\partial\vartheta}. \tag{33}$$

This is readily integrated to yield

$$n^2 = N^2(\Psi) - \frac{L^2}{2m_i TR^2}, \tag{34}$$

where N is a flux function. Once again, we can obtain a relation between the density, the rotation rate and the temperature:

$$n = \frac{N}{\left(1 + m_i\Omega_\zeta^2 R^2/2T\right)^{1/2}}. \tag{35}$$

The three models discussed above illustrate the influence of the torque source distribution and the momentum relaxation time on the density and the angular velocity distributions. We emphasise again that equations (19-21) do not in general imply rigid rotation of flux surfaces, i.e. $\Omega_\zeta = \Omega_\zeta(\Psi)$.

Using the fact that the derivatives with respect to R of flux functions must vanish at the magnetic axis, it is straightforward to show that at this point

$$R\frac{d\ln n}{dR} = \frac{m_i\Omega_\zeta^2 R^2}{2T}, \tag{36}$$

irrespective of whether n is given by equations (27), (28), (31) or (34). Equation (36) is in good agreement with measurements taken at the magnetic axis of plasmas in the National Spherical Torus Experiment (NSTX) [15]. However, it is important to note that such measurements by themselves cannot be used to determine the variation of density on a flux surface.

3.2. Relation between rotation rate and temperature

It is also of interest to eliminate n from the above equations, thereby yielding a direct relation between Ω_ζ and T. Consider, for example, the first of the scenarios considered above, in which $K = m_i n\Omega_\zeta$ is a flux function. Using equation (26) we obtain

$$\Omega_\zeta = \frac{KT^{1/2}}{m_i\left[IT + K^2R^2/2m_i\right]^{1/2}}. \tag{37}$$

Using the ion and electron momentum balance equations, it is possible to obtain expressions for Ω_ζ in terms of T_e and T_i in the limit $F_\zeta^{\rm ext} \to 0$, $\tau_\zeta \to \infty$, $\eta \to 0$,

$V_{\rm L} \to 0$ [6]. In the case of rigidly-rotating flux surfaces the required expression takes the form

$$\Omega_\zeta = \Omega_{\zeta 0} \left(\frac{T}{T_0} \right)^{1/2} \exp \left[- \int_{\Psi_0}^{\Psi} \frac{T_e' d\psi}{4T} \right], \tag{38}$$

where $\Omega_{\zeta 0}$, T_0 and Ψ_0 are constants and $T_e' = dT_e/d\Psi$. When $T_e = T_i$ equation (38) yields

$$\Omega_\zeta = \Omega_{\zeta 0} \left(\frac{T}{T_0} \right)^{1/4}. \tag{39}$$

In this limit the rotation profile is thus predicted to be much flatter than the temperature profile. However, spectroscopic measurements in tokamak plasmas with $T_e \simeq T_i$ indicate that Ω_ζ and T_i generally have fairly similar profiles (see for example figure 5 in [17]), contradicting equation (39). The relation given by equation (37), on the other hand, could be consistent with measured rotation and temperature profiles, depending on the flux functions I and K. Equation (38) can be generalised to allow non-rigid rotation of flux surfaces, while still neglecting sources and sinks of momentum and current [6]:

$$\Omega_\zeta = \frac{AT^{1/2}}{1 + (R/R_0)^2}, \tag{40}$$

where A is a flux function and R_0 is a constant. Equation (40) could also be consistent with measured profiles, depending on A. The important conclusion to be drawn here is that in order to account for measured rotation and temperature profiles it is necessary to invoke either momentum sources and sinks or non-rigid rotation of flux surfaces.

3.3. Ohm's law for tokamak plasma with toroidal flow

We can use the electron momentum balance equation to derive a relation between the loop voltage and the plasma current as follows. Taking the scalar product of equation (12) with $\mathbf{B}$ in the form given by equation (6) and dividing by ne we obtain

$$\frac{J}{ne} \frac{\partial p_e}{\partial \vartheta} = J \frac{\partial \Phi}{\partial \vartheta} - \frac{V_{\rm L} F}{2\pi R^2} + \eta \left[\frac{F j_\zeta}{R} - \frac{B_\vartheta^2}{\mu_0} \frac{\partial F}{\partial \Psi} + \frac{\nabla \Psi \cdot \nabla \vartheta}{\mu_0 R^2} \frac{\partial F}{\partial \vartheta} \right], \tag{41}$$

where B_ϑ is the poloidal magnetic field. When F is a flux function this equation reduces to

$$\frac{J}{ne} \frac{\partial p_e}{\partial \vartheta} = J \frac{\partial \Phi}{\partial \vartheta} - \frac{V_{\rm L} F}{2\pi R^2} + \eta \left[\frac{F j_\zeta}{R} - \frac{F' B_\vartheta^2}{\mu_0} \right]. \tag{42}$$

It follows from equations (8) and (19) that

$$j_\zeta = -R \frac{\partial p}{\partial \Psi} + \frac{1}{2} m_i n \Omega_\zeta^2 R \frac{\partial R^2}{\partial \Psi} - \frac{FF'}{\mu_0 R}. \tag{43}$$

Substituting this expression into equation (42) and dividing by F we obtain

$$\frac{J}{neF}\frac{\partial p_e}{\partial\vartheta} = \frac{J}{F}\frac{\partial\Phi}{\partial\vartheta} - \frac{V_{\rm L}}{2\pi R^2} - \eta\left(\frac{\partial p}{\partial\Psi} - \frac{1}{2}m_i n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi} - \frac{F'B^2}{F}\right). \quad (44)$$

Putting $p_e = nT_e$, assuming that η and T_e are flux functions, and averaging equation (44) over ϑ, we obtain

$$\frac{V_{\rm L}}{2\pi}\langle\frac{1}{R^2}\rangle = -\eta\left(\langle p\rangle' - \frac{1}{2}m_i\langle n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi}\rangle + \frac{F'}{\mu_0 F}\langle B^2\rangle\right), \quad (45)$$

and hence

$$FF' = -\frac{\mu_0 F^2}{\langle B^2\rangle}\left[\frac{V_{\rm L}}{2\pi\eta}\langle\frac{1}{R^2}\rangle + \langle p\rangle' - \frac{1}{2}m_i\langle n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi}\rangle\right]. \quad (46)$$

Substituting this expression into equation (43) we obtain

$$j_\zeta = -R\left(\frac{\partial p}{\partial\Psi} - \frac{F^2\langle p\rangle'}{R^2\langle B^2\rangle}\right) + \frac{1}{2}m_i R\left(n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi} - \frac{F^2\langle n\Omega_\zeta^2\partial R^2/\partial\Psi\rangle}{R^2\langle B^2\rangle}\right) + \frac{F^2 V_{\rm L}}{2\pi R\eta}\frac{\langle 1/R^2\rangle}{\langle B^2\rangle}. \quad (47)$$

Now the total plasma current is given by the expression

$$I_p = \iint_A j_\zeta dRdZ = \int_{\Psi_0}^0 \frac{d\Psi}{J}\int_0^{2\pi}\frac{j_\zeta d\vartheta}{R} = 2\pi\int_{\Psi_0}^0\langle\frac{j_\zeta}{R}\rangle\frac{d\Psi}{J}, \quad (48)$$

where A is the plasma poloidal cross-section area, $\Psi_0(<0)$ is the poloidal flux at the magnetic axis, and we are using the convention that Ψ vanishes at the plasma edge. Substituting the expression for j_ζ given by equation (47) into equation (48), we deduce that the loop voltage $V_{\rm L}$ and plasma current I_p are related by an Ohm's law of the form

$$I_p = 2\pi\int_{\Psi_0}^0\left(1 - \frac{F^2\langle 1/R^2\rangle}{\langle B^2\rangle}\right)\left(\frac{1}{2}m_i\langle n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi}\rangle - \langle p\rangle'\right)\frac{d\Psi}{J} + V_{\rm L}\int_{\Psi_0}^0\frac{F^2\langle 1/R^2\rangle^2}{\langle B^2\rangle}\frac{d\Psi}{\eta J}. \quad (49)$$

Using $F^2\langle 1/R^2\rangle = \langle B_\zeta^2\rangle$ and $\langle B_\zeta^2\rangle + \langle B_\vartheta^2\rangle = \langle B^2\rangle$, we may write equation (49) in a somewhat more transparent form:

$$I_p = 2\pi\int_{\Psi_0}^0\frac{\langle B_\vartheta^2\rangle}{\langle B^2\rangle}\left(\frac{1}{2}m_i\langle n\Omega_\zeta^2\frac{\partial R^2}{\partial\Psi}\rangle - \langle p\rangle'\right)\frac{d\Psi}{J} + V_{\rm L}\int_{\Psi_0}^0\frac{\langle B_\zeta^2\rangle\langle 1/R^2\rangle}{\langle B^2\rangle}\frac{d\Psi}{\eta J}. \quad (50)$$

Equation (49) or equation (50) could be used to calculate the loop voltage required to maintain a specified plasma current, given the toroidal rotation profile together with radial profiles of η, J and the flux surface-averaged quantities $\langle B_\zeta^2\rangle$, $\langle B^2\rangle$ and $\langle 1/R^2\rangle$. In the large aspect ratio limit and in the absence of toroidal flows it is evident that the ratio of the first integral on the right hand side of equation (50) to the second integral is of order $\epsilon p\eta/(q^2 B_\vartheta V_{\rm L})$ where $\epsilon \ll 1$ is inverse aspect ratio. Assuming neoclassical resistivity and typical parameter values for plasmas in the JET tokamak [17] ($\epsilon \sim 0.1$, $n \sim 4 \times 10^{19}\,\mathrm{m}^{-3}$, $T_i \sim T_e \sim 3\,\mathrm{keV}$, $q \sim 1$, $B_\vartheta \sim 0.3\,\mathrm{T}$, $V_\mathrm{L} \simeq 0.5\,\mathrm{V}$) we find that this ratio is around 3×10^{-4}. Hence in conventional tokamaks the right hand side of equation (50) is dominated by the loop voltage term. The relative contribution of the term involving $\langle B_\vartheta^2 \rangle$ is somewhat higher in spherical tokamaks, but is still much less than unity.

It is important to note that our two-fluid model does not include trapped particle effects, and therefore the bootstrap current contribution to I_p [18] is not taken into account in equations (49) and (50). Moreover we have assumed isotropic scalar resistivity η, whereas collisional transport theory indicates that parallel electrical conductivity in a plasma is approximately equal to twice the perpendicular conductivity [19]. The Ohm's law obtained above is nevertheless physically illuminating, providing as it does a simple relation between V_L and I_p, and could be applied quantitatively in global tokamak fluid codes such as `CENTORI` that employ scalar resistivity [10].

We address finally the question of calculating self-consistently the equilibrium radial electric field. Taking the scalar product of the ion momentum balance equation (10) with $\nabla\vartheta \times \nabla\zeta$, assuming again that $\mathbf{F}^{\mathrm{ext}}$ and $\mathbf{F}^{\mathrm{drag}}$ are in the toroidal direction, and dividing by neJ, we obtain

$$\frac{\partial \Phi}{\partial \Psi} = -\frac{1}{en}\frac{\partial p_i}{\partial \Psi} - \Omega_\zeta + \frac{m_i \Omega_\zeta^2}{2e}\frac{\partial R^2}{\partial \Psi} - \eta \mathbf{j} \cdot \frac{\nabla\vartheta \times \nabla\zeta}{J}. \qquad (51)$$

It is straightforward to verify that the last term on the right hand side is negligible compared with the other terms under typical tokamak conditions. Averaging the remaining terms with respect to ϑ we obtain a simple expression for $\partial\langle\Phi\rangle/\partial\Psi$, which is a measure of the flux surface-averaged radial electric field. Such fields can be determined in tokamak plasmas from motional Stark effect measurements [20], and their shear is thought to play an important role in transport barrier physics. The equilibrium electric field given by the flux surface-averaged form of equation (51) could thus be compared directly with experiment.

4. Conclusions

We have extended previous analyses of axisymmetric plasma equilibria with toroidal flows to take into account the fact that such flows must be continuously driven when dissipative effects are present, as they invariably are in the case of tokamak plasmas. For this purpose it is convenient to use plasma-based coordinates such that the Jacobian of the transformation from laboratory cartesian coordinates to plasma coordinates is a function of poloidal flux, Ψ (rather than being a strict constant, as in the case of Hamada coordinates). Using such a coordinate system, we have shown that the steady-state ion and electron fluid momentum balance equations yield a set of three coupled partial differential equations describing the spatial variation of Ψ, plasma pressure p

and toroidal magnetic field B_ζ. These equations demonstrate explicitly that there is no *a priori* reason to suppose that magnetic flux surfaces in axisymmetric plasmas should rotate as rigid bodies, as required by ideal MHD in the absence of momentum sources and poloidal flows. For several specific assumed relations between momentum drive and dissipation, we have derived expressions for the variation of density on flux surfaces that differ from results obtained in the dissipationless limit. We have also obtained a relation between temperature and toroidal rotation rate, and a relation between the loop voltage and plasma current in an inductive tokamak plasma with toroidal flow. The latter result could be used to determine the loop voltage required to maintain a particular plasma current in a tokamak discharge with slowly-evolving field and plasma profiles. We have also determined a simple expression for the equilibrium radial electric field which can be compared directly with experiment.

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